

Validation of Symmetry Induced Bifurcation Points in Nonlinear Diffusion Problems

Maxime Breden*

Evelyn Sander[†]

Thomas Wanner[‡]

September 29, 2026

Abstract

Nonlinear parabolic partial differential equations frequently occur in the modeling of applied phenomena. They often exhibit complicated dynamics which is organized by their associated equilibrium solutions. As a function of the involved model parameters, the equilibria can be detected using a bifurcation analysis, which in turn is centered around a deeper understanding of bifurcation points. In many applied systems establishing these organizing centers rigorously is difficult to achieve using traditional mathematical techniques. In this paper, we propose a theoretical framework for computer-assisted proofs of symmetry-induced transcritical and pitchfork bifurcations in nonlinear problems. This is accomplished by reformulating the existence question via a zero finding problem for an associated extended nonlinear system. The approach is general and allows for a wide variety of underlying symmetries. We demonstrate how these results can be applied in the setting of parabolic systems and higher-order equations, on both one- and two-dimensional domains.

AMS subject classifications: Primary: 37G40, 37M20, 65G20, 65P30; Secondary: 37B35, 37C81, 65G30, 74G60, 74N15.

Keywords: Bifurcations, nonlinear diffusion, symmetry-breaking, transcritical and pitchfork bifurcations, computer-assisted proofs, interval arithmetic, rigorous validation

*CMAP, CNRS, École polytechnique, Institut Polytechnique de Paris, 91120 Palaiseau, France. (maxime.breden@polytechnique.edu).

[†]Department of Mathematical Sciences, George Mason University, Fairfax, Virginia 22030, USA (esander@gmu.edu).

[‡]Department of Mathematical Sciences, George Mason University, Fairfax, Virginia 22030, USA (twanner@gmu.edu).

Contents

1	Introduction	2
2	A general bifurcation result	7
2.1	Classical bifurcation from a simple eigenvalue	8
2.2	A sufficient condition for symmetry-breaking bifurcation	9
2.3	Detecting bifurcations via extended systems	14
2.4	Verifying the symmetry assumptions	17
3	Results	18
4	Computer-assisted proof techniques	21
4.1	The Newton-Kantorovich approach	23
4.2	Application to the Ohta-Kawasaki model	26
4.3	Application to the SKT system	30
5	Determining the bifurcation type	32
5.1	Establishing bifurcation symmetries	32
5.2	Proving transcritical bifurcations	35
A	Computable bounds for the Ohta-Kawasaki equation	38
B	Computable bounds for the SKT equation	41
C	Computable bounds for proving transcriticality	42
	References	44

1 Introduction

Bifurcation theory has long been used to study the zeros of nonlinear parameter-dependent operators, see for example [8, 17] and the references therein. As the underlying parameters change, new zeros can only occur via bifurcation points, which therefore act as organizing centers for the global structure of the zero set. In view of Sard’s theorem, in general nonlinear problems only saddle-node bifurcations can be observed in a perturbation-robust sense, see for example the description in [45]. The situation is different if the nonlinear problem under consideration is for example generated by the study of equilibrium solutions of parabolic partial differential equations. In this case, perturbations of the differential equation lead to very specific perturbations of the associated nonlinear zero finding problem, and they can and often will exhibit transcritical and pitchfork bifurcation points which are robust under parameter changes. Usually, this robustness is a consequence of some symmetry which is preserved by the differential equation and some of its stationary states, and which is then broken at these bifurcation points — hence the name *symmetry-breaking bifurcation*.

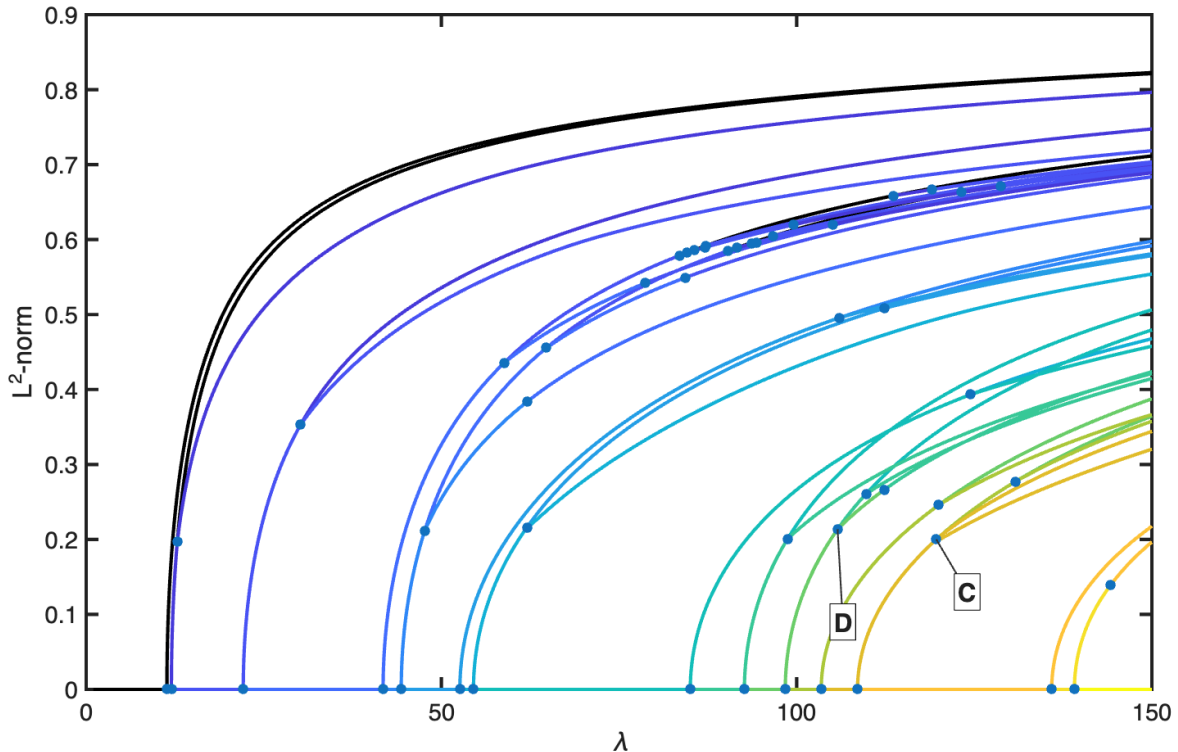


Figure 1: A portion of the bifurcation diagram of steady states for the Ohta–Kawasaki model (1) in two dimensions, for total mass $\mu = -0.1$, interaction parameter $\sigma = 1$, and a slightly rectangular domain $\Omega = (0, 1) \times (0, 0.97)$ to avoid extra symmetries. The pieces of branches are colored by Morse index, and the blue dots represent potential pitchfork and transcritical bifurcations which have been detected using the AUTO software package. The ones labeled C and D are validated in Theorem 3.1. This figure is by no means a complete picture.

While the general mathematical theory of symmetry-breaking bifurcations is well-developed [7, 10], applying these results to specific applied models can be challenging. Usually, one can only study bifurcation points which are on “simple” solution branches directly, such as branches of spatially constant solutions. As soon as the focus shifts towards secondary bifurcations from nontrivial solution branches, pen-and-paper existence proofs frequently are no longer feasible.

In such situations computer-assisted proof techniques can fill the void. It was shown in [22] that both saddle-node and standard $\mathbb{Z}_2$ -symmetry breaking pitchfork bifurcations can be rigorously established in the Ohta–Kawasaki model for the formation of diblock copolymers. This work was based on the numerical studies in [15], and it was accomplished by reformulating both bifurcation types as zero finding problems for suitable extended nonlinear systems. These extended systems contain not only the equilibrium condition, but also force the existence of a zero eigenvalue of the Fréchet derivative at the bifurcation point.

While nominally the extended systems for these two situations look the same, the distinction between the two types of bifurcation points is achieved via domain restrictions for the underlying operators. Based on this reformulation, it was possible in [22] to apply standard computer-assisted proof techniques to rigorously verify branches of saddle-node and $\mathbb{Z}_2$ -symmetry breaking bifurcation points in a two-parameter setting.

The results in [22] led to the validation of many pitchfork bifurcation points in the Ohta–Kawasaki equilibrium model on one-dimensional domains. However, numerical studies of this model also uncovered numerous potential bifurcation points which could not be treated in this way. In these cases, the underlying symmetries were more complicated in nature, induced by cyclic groups of higher order. In [30] we were able to extend the previous approach to cover these cases as well, by suitably changing the extended system. The computer-assisted proofs in that paper were modeled after the approach described in [29, 32, 33, 44, 46], but they relied heavily on the specific form of the underlying cyclic symmetry operator. Common to both of the results described so far was the exclusive focus on saddle-node and pitchfork bifurcations, the assumption of zero mass in the Ohta–Kawasaki model, and the restriction to one-dimensional base domains.

In the present paper, we seek to generalize the above-mentioned results in several significant directions. These can be described as follows:

- Move beyond the simple $\mathbb{Z}_2$ and cyclic symmetries considered so far by describing an approach which in principle allows for arbitrary general symmetries. In particular, shift the emphasis from detailed knowledge of the involved symmetry operators to a formulation which focuses on the underlying symmetry subspace only.
- Derive methods for dealing with transcritical bifurcations, which have for example been observed numerically in the Ohta–Kawasaki model for nonzero mass.
- Develop an approach that allows one to also consider partial differential equations on higher-dimensional domains as well as to extend the class of equation types that can be considered.

All of the above points are achieved using the results of this paper, while still employing the appropriate extended systems. In fact, we can describe in more detail how the extended systems have to be adjusted based on the underlying symmetries in the system. Moreover, by restricting attention to a reformulation in terms of a zero finding problem, one can then establish the existence of the bifurcation point using a variety of different available computer-assisted proof techniques.

This paper is part of a broader body of works which develop computer-assisted techniques to study bifurcations, and an approach similar to the one described here can be used to study cusp bifurcations [21]. Also computer-assisted, but slightly different in spirit, one can rigorously study bifurcations using desingularization, see for example [39] for Hopf bifurcations, or quantitative Lyapunov-Schmidt reduction and normal form calculations [1, 47]. Going even further, a complete description of the dynamics near the bifurcation can sometimes be obtained by computer-assisted means [18].

The main results of this paper are illustrated in the context of two specific partial differential equations. First, we consider the previously-mentioned Ohta–Kawasaki equation

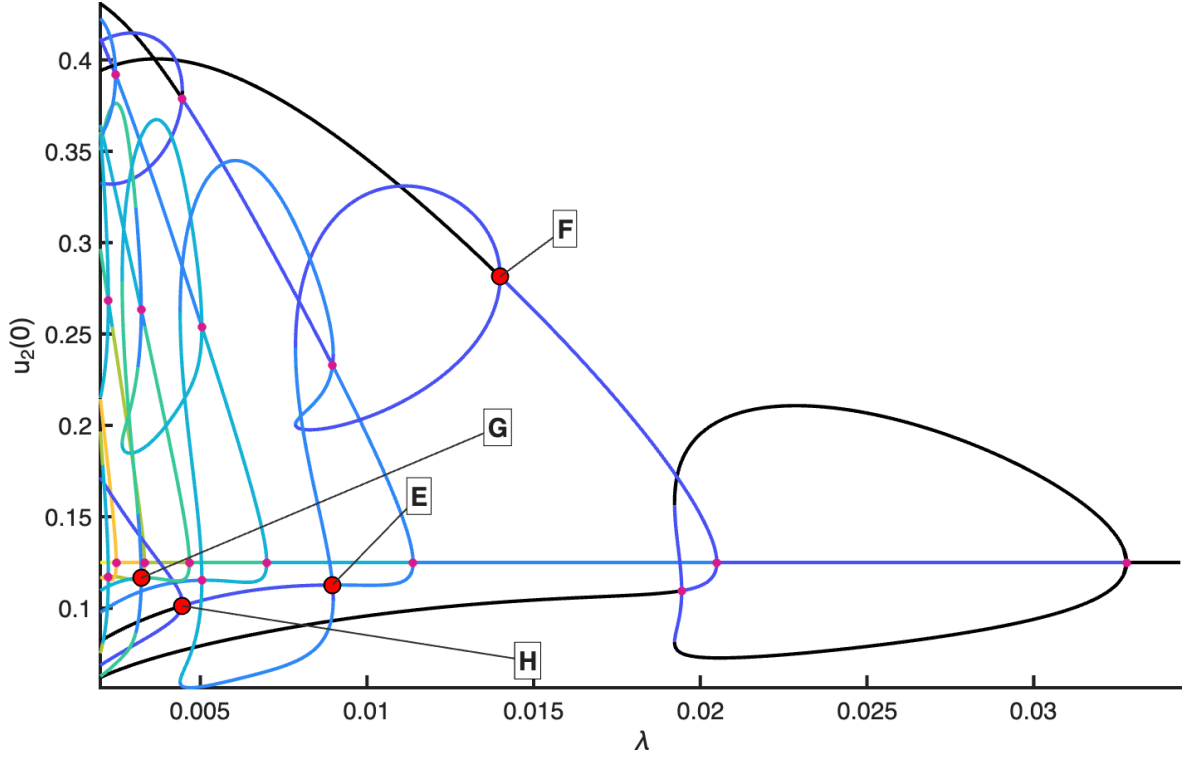


Figure 2: A portion of the bifurcation diagram of steady states for the SKT model (2) on the domain $\Omega = (0, 1)$ with parameters (3), first obtained in [13]. To be consistent with this earlier work, the vertical axis represents the value $u_2(0)$ of the second component at the left boundary. The red dots represent potential pitchfork and transcritical bifurcations which have been detected using the AUTO software package. The ones labeled E to H are validated in Theorem 3.2. This figure is by no means a complete picture of all possible bifurcations.

describing phase separation in diblock copolymers. This model is given by the parabolic partial differential equation

$$\partial_t u = -\Delta (\Delta u + \lambda (u - u^3)) - \lambda \sigma (u - \mu) \quad \text{in } \Omega, \quad (1)$$

subject to homogeneous Neumann boundary conditions for both u and Δu . While the parameter λ serves as the main bifurcation parameter, the model also depends on the total mass μ and the polymer interaction parameter σ . We will be able to validate symmetry-breaking bifurcations in the equilibrium structure of this model for the case of nonzero mass. In addition, we rigorously establish transcritical bifurcation points which persist under parameter variation for the first time. Finally, existence proofs for bifurcation points on two-dimensional rectangular domains are included as well, as shown for instance in Figure 3.

The second problem we study in this paper is the Shigesada–Kawasaki–Teramoto (SKT) cross diffusion system, a quasi-linear parabolic system introduced in [35] to model competing

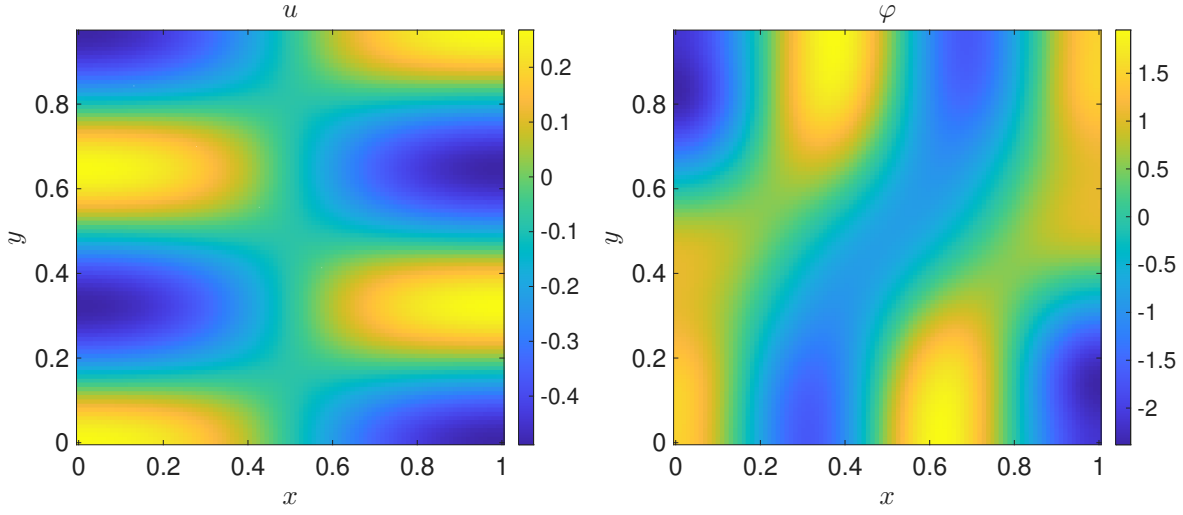


Figure 3: An approximate steady state solution u of the Ohta–Kawasaki equation (1) for the parameters $\mu = -0.1$, $\sigma = 1$, and $\Omega = (0, 1) \times (0, 0.97)$, which undergoes a symmetry-breaking transcritical bifurcation at $\lambda \approx 119.66$, with approximate eigenfunction φ . This bifurcation is labeled C in Table 1. The occurrence of this bifurcation is rigorously established in Theorem 3.1.

species

$$\begin{aligned} \partial_t u_1 &= \Delta (d_1 u_1 + d_{11} u_1^2 + d_{12} u_1 u_2) + (r_1 u_1 - a_1 u_1^2 - b_1 u_1 u_2) \\ \partial_t u_2 &= \Delta (d_2 u_2 + d_{21} u_1 u_2 + d_{22} u_2^2) + (r_2 u_2 - a_2 u_2^2 - b_2 u_1 u_2) \end{aligned} \quad \text{in } \Omega, \quad (2)$$

subject to homogeneous Neumann boundary conditions. The presence of cross-diffusion terms in this system provides a potentially destabilizing mechanism akin to a Turing instability [14], which can lead to pattern formation, and intricate bifurcation diagrams of steady states have indeed been obtained numerically for this model [4, 13, 19], as also shown in Figure 2. Following previous studies, we take $d_1 = d_2 = \lambda$ as the bifurcation parameter, while the other parameters are fixed as

$$d_{11} = d_{21} = d_{22} = 0, \quad d_{12} = 3, \quad r_1 = 5, \quad r_2 = 2, \quad a_1 = a_2 = 3, \quad b_1 = b_2 = 1. \quad (3)$$

For this system as well, we validate a variety of symmetry-breaking pitchfork and transcritical bifurcation points, for one- and two-dimensional domains, one of which is illustrated in Figure 4. In fact, we will see that while in some cases the numerical approximations do not provide a clear answer as to which type of bifurcation point is observed, the computer-assisted proof result resolves the question immediately.

The remainder of this paper is organized as follows. In Section 2 we describe the main theoretical results necessary for our approach. It is based on a general bifurcation result in the sense of Crandall–Rabinowitz, which provides precise conditions for the occurrence of a transcritical or pitchfork bifurcation. In contrast to earlier results, the required assumptions focus on the actual symmetry subspace of the primary branch, rather than the equivariance

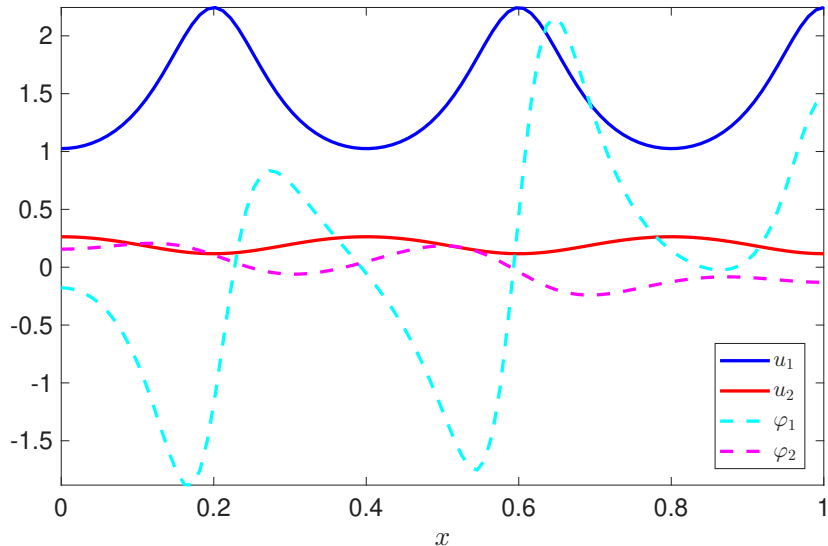


Figure 4: An approximate steady state (u_1, u_2) of the SKT system (2) with $\Omega = (0, 1)$ and parameters as in (3), which undergoes a symmetry-breaking pitchfork bifurcation at $\lambda \approx 3.227 \times 10^{-3}$, with approximate eigenfunction (φ_1, φ_2) . This bifurcation is labeled G in Table 2. The occurrence of this bifurcation is rigorously established in Theorem 3.2.

operators describing its symmetry. In addition, we show that this reformulation naturally allows us to use a suitable extended system combined with a zero finding method to establish the respective bifurcation point. This result is formulated in general Banach spaces to make it as applicable as possible, and we also discuss how its assumptions can be verified in practice. In Section 3 we illustrate our approach by presenting new bifurcation results for both the Ohta–Kawasaki equation and the SKT system in dimensions one and two. The computer-assisted approach underlying these results is presented in Section 4, where we explain how to rigorously validate zeros of the extended system introduced in Section 2. Finally the additional work required to determine whether the obtained bifurcations are of transcritical or pitchfork type is described in Section 5.

2 A general bifurcation result

In this section we present a general bifurcation result which can be used to establish the existence of transcritical or pitchfork bifurcation points in a variety of symmetry-breaking settings. It is based on the celebrated Crandall–Rabinowitz result [9], which for the sake of completeness is recalled in Section 2.1. Our new result is the subject of Section 2.2. In this section we isolate the crucial framework necessary for symmetry-breaking bifurcations with a focus on the underlying symmetry space, rather than focusing on the equivariance operators. In Section 2.3, we continue in the spirit of [22, 30] and demonstrate how these symmetry-breaking bifurcations can be detected via root-finding in a suitable extended system. The result contains as special cases both previously-developed cases: The classical $\mathbb{Z}_2$ -symmetry

breaking [22] and the more general cyclic symmetry results of [30]. In contrast to the previous results, this more general framework opens the door to straightforward applications for partial differential equations on higher-dimensional domains. Finally, in Section 2.4 we indicate how the main assumption of our bifurcation result can be validated in practice, thereby establishing its versatility.

2.1 Classical bifurcation from a simple eigenvalue

One of the central results in early bifurcation theory is the Crandall-Rabinowitz theorem on bifurcations from simple eigenvalues [9]. We briefly recall it in this section, in order to be able to contrast the classical theory with our new result given in the next section. In the stated form, the theorem is the Banach space analogue of the finite-dimensional version given in [34, Theorem 5.28, p. 707].

Theorem 2.1 (Secondary bifurcation from a simple eigenvalue). *For Banach spaces X and Y consider a smooth function $F : \mathbb{R} \times X \rightarrow Y$, as well as the associated zero finding problem*

$$F(\lambda, u) = 0 \quad \text{for} \quad (\lambda, u) \in \mathbb{R} \times X . \quad (4)$$

Let $p : I \rightarrow X$, where $I \subset \mathbb{R}$ is an open interval, denote a smooth function which satisfies

$$F(\lambda, p(\lambda)) = 0 \quad \text{for all} \quad \lambda \in I \subset \mathbb{R} ,$$

i.e., it establishes a primary solution branch of (4). Suppose further that for some $\lambda_o \in I$ and $u_o = p(\lambda_o)$ the Fréchet derivative $L = D_u F(\lambda_o, u_o)$ is a Fredholm operator of index zero which has a simple eigenvalue zero. That is, we assume that the dimension of the nullspace of L is 1 and that there is a nonzero element $\varphi_o \in X$ such that $N(L) = \text{span}[\varphi_o]$. Let $\psi_o^ \in Y^*$ denote any element of the dual space of Y for which $R(L) = N(\psi_o^*)$. Finally, suppose that we have*

$$D_{\lambda u} F(\lambda_o, u_o)[\varphi_o] + D_{uu} F(\lambda_o, u_o)[\varphi_o, p'(\lambda_o)] \notin R(D_u F(\lambda_o, u_o)) . \quad (5)$$

Then the point (λ_o, u_o) is a bifurcation point for the nonlinear problem (4). Additionally, the precise bifurcation type can be determined as follows:

(a) *If one has in addition that*

$$D_{uu} F(\lambda_o, u_o)[\varphi_o, \varphi_o] \notin R(D_u F(\lambda_o, u_o)) , \quad (6)$$

then the point (λ_o, u_o) is a transcritical bifurcation point.

(b) *On the other hand, if we have*

$$D_{uu} F(\lambda_o, u_o)[\varphi_o, \varphi_o] \in R(D_u F(\lambda_o, u_o)) , \quad (7)$$

as well as

$$\varrho_o = \frac{\psi_o^* D_{uuu} F(\lambda_o, u_o)[\varphi_o, \varphi_o, \varphi_o] + 3\psi_o^* D_{uu} F(\lambda_o, u_o)[\varphi_o, \zeta_o]}{3\psi_o^* D_{\lambda u} F(\lambda_o, u_o)[\varphi_o] + 3\psi_o^* D_{uu} F(\lambda_o, u_o)[\varphi_o, p'(\lambda_o)]} \neq 0 ,$$

where $\zeta_o \in X$ is any element satisfying

$$D_{uu}F(\lambda_o, u_o)[\varphi_o, \varphi_o] + D_uF(\lambda_o, u_o)[\zeta_o] = 0 ,$$

then the point (λ_o, u_o) is a pitchfork bifurcation point. If ϱ_o is strictly positive, then the bifurcating solutions exist for $\lambda < \lambda_o$ close to the bifurcation point, while for negative ϱ_o they exist for $\lambda > \lambda_o$. In the case $\varrho_o = 0$ the local shape of the bifurcating branch cannot be determined conclusively, yet the bifurcating branch is still orthogonal to the parameter direction at the bifurcation point.

The above theorem is remarkable in its simplicity. By just assuming the existence of a solution branch $u = p(\lambda)$, and a noninvertible Fréchet derivative at a point on this branch with simple zero eigenvalue, as well as the generically satisfied nondegeneracy condition (5), the result guarantees a bifurcating solution branch. Parts (a) and (b) are basically just window dressing in that they establish the nature of this new branch in more detail. Nevertheless, while this is a remarkable *theoretical* result, its usefulness hinges on knowledge of the primary solution branch $u = p(\lambda)$ — and this fact severely restricts its applicability in concrete situations. The only exception to this comment is an existing *trivial solution branch* $p \equiv 0$, and in fact, in many texts the above result is only formulated for this special situation.

In the context of computer-assisted proofs, Theorem 2.1 poses challenges. The primary branch is usually only known in an approximate way, and therefore special care has to be taken to obtain tight enough error estimates so that the nondegeneracy condition can be verified at the (unknown) precise bifurcation point location. While this can be achieved in many cases, see [16] for an example, it becomes significantly more difficult if one is interested in rigorously establishing branches of bifurcation points as in [22]. Consequently, in our previous work [22, 30] we only considered the Ohta–Kawasaki model with zero total mass and odd nonlinearity, which in turn provided additional information on the primary branch. Essentially, this allowed us to consider a problem with trivial solution after a suitable transformation. Yet, this special situation also implied the automatic validity of (7), which explains why these papers only considered the case of pitchfork bifurcations.

2.2 A sufficient condition for symmetry-breaking bifurcation

In order to turn Theorem 2.1 into a result which is more flexible and general but also amenable to the use of computer-assisted proofs, it needs to be reformulated in such a way that the existence of the primary branch does not need to be verified separately, and simply becomes a part of the theorem, with conditions which can be checked at a single fixed parameter value. This will be accomplished in the present section.

We begin by presenting a modified sufficient condition that ensures the occurrence of a transcritical or a pitchfork bifurcation in nonlinear problems. Rather than considering the generality of the above Crandall–Rabinowitz result, we now restrict our attention to the situation of *symmetry-breaking*. For our result below, we need two assumptions, which we now state.

The first part of Assumption 2.2 formalizes the statement that at the point (λ_o, u_o) the implicit function theorem fails, and therefore this point has the potential to be a bifurcation

point. The additional parts of Assumption 2.2 give rise to a situation which stays close to the one of finite-dimensional linear algebra. It is given as follows.

Assumption 2.2 (Necessary condition for bifurcation). *Let X and Y denote two Banach spaces, and suppose that $F : \mathbb{R} \times X \rightarrow Y$ is a smooth nonlinear operator. Furthermore, assume that there exists a pair $(\lambda_o, u_o) \in \mathbb{R} \times X$ with*

$$F(\lambda_o, u_o) = 0 , \quad (8)$$

and that the Fréchet derivative $L = D_u F(\lambda_o, u_o) \in \mathcal{L}(X, Y)$ is a Fredholm operator of index zero. Finally, suppose that the nullspace of L is one-dimensional and that we have both

$$N(L) = \text{span}[\varphi_o] \quad \text{and} \quad R(L) = N(\psi_o^*) \quad (9)$$

for some nonzero elements $\varphi_o \in X$ and $\psi_o^ \in Y^*$, where Y^* is the dual space of Y .*

While Assumption 2.2 above describes the general framework that is necessary for the Crandall-Rabinowitz approach, Assumption 2.3 below focuses specifically on the aspect of symmetry-breaking. In particular, we assume the existence of closed symmetry subspaces X_{sym} and Y_{sym} which are preserved by F , along with a Fredholm condition on how L acts on these spaces.

Assumption 2.3 (Symmetry-breaking). *Suppose that Assumption 2.2 is satisfied, and that there exist closed symmetry subspaces $X_{\text{sym}} \subset X$ and $Y_{\text{sym}} \subset Y$ with $F(\mathbb{R}, X_{\text{sym}}) \subset Y_{\text{sym}}$. In addition, assume that the restriction $L|_{X_{\text{sym}}} : X_{\text{sym}} \rightarrow Y_{\text{sym}}$ is again a Fredholm operator of index zero, and that we have both*

$$u_o \in X_{\text{sym}} \quad \text{and} \quad \varphi_o \notin X_{\text{sym}} . \quad (10)$$

We would like to emphasize that an especially crucial aspect of Assumption 2.3 is the fact that the restriction $L|_{X_{\text{sym}}} : X_{\text{sym}} \rightarrow Y_{\text{sym}}$ is again a Fredholm operator of index zero. Note that this is by no means automatic. For example, there are situations where one could choose a finite-dimensional X_{sym} together with $Y_{\text{sym}} = Y$, and in the case of infinite-dimensional Y this would violate the finite codimension of $L(X_{\text{sym}})$. Even if the restricted operator $L|_{X_{\text{sym}}} : X_{\text{sym}} \rightarrow Y_{\text{sym}}$ is Fredholm, one can easily come up with examples in which its index is nonzero, by choosing the space Y_{sym} too large. Nevertheless, as we will elaborate in Section 2.4, in our applications the verification of Assumption 2.3 amounts to little more than considering the differential operator describing L for functions which are defined on a suitable subdomain of Ω .

Notice also that Assumption 2.3 does not rely on a priori knowledge of a primary solution branch for the nonlinear problem $F(\lambda, u) = 0$. In fact, its formulation enables us to prove the existence of such a branch as part of the theorem, using only information at a single point (λ_o, u_o) . In this way, the more specific requirements in Assumptions 2.2 and 2.3 remove one of the main disadvantages of Theorem 2.1.

The following result confirms that the combination of the above two assumptions provides a sufficient condition for the existence of a transcritical or pitchfork bifurcation. In fact, the only additional requirement is a mild nondegeneracy condition.

Theorem 2.4 (Symmetry breaking bifurcations). *Suppose that Assumptions 2.2 and 2.3 are satisfied, i.e., in particular we have $F(\lambda_o, u_o) = 0$, $L\varphi_o = D_u F(\lambda_o, u_o)[\varphi_o] = 0$, the nullspace of L is one-dimensional, $u_o \in X_{\text{sym}}$, and $\varphi_o \notin X_{\text{sym}}$. Under these assumptions, there exists a unique function $\xi_o \in X_{\text{sym}}$ such that*

$$L\xi_o + D_\lambda F(\lambda_o, u_o) = 0. \quad (11)$$

Under the further assumption that the nondegeneracy condition

$$D_{\lambda u} F(\lambda_o, u_o)[\varphi_o] + D_{uu} F(\lambda_o, u_o)[\varphi_o, \xi_o] \notin R(L) \quad (12)$$

is satisfied, the point (λ_o, u_o) is a bifurcation point, and the following hold:

- (a) *If $D_{uu} F(\lambda_o, u_o)[\varphi_o, \varphi_o] \notin R(L)$, then there is a transcritical bifurcation at (λ_o, u_o) .*
- (b) *If $D_{uu} F(\lambda_o, u_o)[\varphi_o, \varphi_o] \in R(L)$, then there is a pitchfork bifurcation at (λ_o, u_o) .*

In both cases, the bifurcating solution branch is not a subset of the symmetry space X_{sym} . That is, the point (λ_o, u_o) is a symmetry-breaking bifurcation point.

Proof. We first show that (11) has a unique solution $\xi_o \in X_{\text{sym}}$. To see this, note that the inclusion $F(\mathbb{R}, X_{\text{sym}}) \subset Y_{\text{sym}}$ and $u_o \in X_{\text{sym}}$ immediately imply $D_\lambda F(\lambda_o, u_o) \in Y_{\text{sym}}$. Due to $\varphi_o \notin X_{\text{sym}}$ the restriction $L|_{X_{\text{sym}}} : X_{\text{sym}} \rightarrow Y_{\text{sym}}$ is one-to-one. Since this restricted operator is a one-to-one index zero Fredholm operator, it is also onto. This establishes the existence of a solution $\xi_o \in X_{\text{sym}}$ solving (11). Since the difference of any two such solutions is contained in $N(L) \cap X_{\text{sym}} = \{0\}$, the uniqueness follows.

The proof of the second part of the theorem is based on a standard Lyapunov-Schmidt reduction argument. Let $\tilde{Y}$ denote an arbitrary one-dimensional subspace of Y such that $Y = R(L) \oplus \tilde{Y}$. In view of $\varphi_o \notin X_{\text{sym}}$, there is a closed subspace $\tilde{X} \subset X$ such that both $X_{\text{sym}} \subset \tilde{X}$ and $X = N(L) \oplus \tilde{X}$ are satisfied. Such a space can be defined as $\tilde{X} = N(\psi^*)$, where $\psi^* \in X^*$ is any functional which vanishes on X_{sym} and is nontrivial on $N(L) = \text{span}[\varphi_o]$, and which exists due to the Hahn-Banach theorem. Finally, let $P : Y \rightarrow Y$ and $Q : X \rightarrow X$ denote the continuous projectors with $R(P) = \tilde{Y}$ and $N(P) = R(L)$, as well as $R(Q) = N(L)$ and $N(Q) = \tilde{X}$. In other words, both projectors have rank one.

The properties of P and Q imply that for any $u \in X$ the elements $v = Qu$ and $w = (I - Q)u$ satisfy $u = v + w \in N(L) \oplus \tilde{X}$. Furthermore, the nonlinear equation $F(\lambda, u) = 0$ is equivalent to the system

$$PF(\lambda, v + w) = 0 \quad \text{and} \quad G(\lambda, v, w) := (I - P)F(\lambda, v + w) = 0, \quad (13)$$

where $G : \mathbb{R} \times N(L) \times \tilde{X} \rightarrow R(L)$. If we write $u_o = v_* + w_* \in N(L) \oplus \tilde{X}$, then $u_o \in X_{\text{sym}} \subset \tilde{X}$ yields both $v_* = 0$ and $w_* = u_o$. Moreover, the Fréchet derivative $D_w G(\lambda_o, 0, u_o)$ satisfies

$$D_w G(\lambda_o, 0, u_o) = (I - P)D_u F(\lambda_o, u_o)|_{\tilde{X}} \in \mathcal{L}(\tilde{X}, R(L)).$$

The fact that $N(L) \cap \tilde{X} = \{0\}$ implies that this Fréchet derivative is invertible. An application of the implicit function theorem guarantees the existence of open neighborhoods $\lambda_o \in \Lambda_0 \subset \mathbb{R}$

and $0 \in V_0 \subset N(L)$, as well as $u_o \in \tilde{X}_0 \subset \tilde{X}$, and a smooth function $W : \Lambda_0 \times V_0 \rightarrow \tilde{X}_0$, such that for all $(\lambda, v) \in \Lambda_0 \times V_0$ one has

$$G(\lambda, v, W(\lambda, v)) = (I - P)F(\lambda, v + W(\lambda, v)) = 0, \quad (14)$$

and such that $W(\lambda, v) \in \tilde{X}_0$ is the unique value in $\tilde{X}_0$ with this property. Note that this implies in particular the identity $W(\lambda_o, 0) = u_o$.

In order to solve the full original problem $F(\lambda, u) = 0$, one simultaneously also has to solve the first equation in (13). Since $N(L)$ is one-dimensional, every $v \in N(L)$ can be uniquely written as $\alpha\varphi_o$, where $\alpha \in \mathbb{R}$. Now consider the bifurcation function $b : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined via

$$b(\lambda, \alpha) := \psi_o^* P F(\lambda, \alpha\varphi_o + W(\lambda, \alpha\varphi_o)), \quad (15)$$

where ψ_o^* was introduced in (9). Then one has $F(\lambda, v + w) = 0$ for $(\lambda, v, w) \in \Lambda_0 \times V_0 \times \tilde{X}_0$ if and only if both $b(\lambda, \alpha) = 0$ and $u = v + w = \alpha\varphi_o + W(\lambda, \alpha\varphi_o)$ are satisfied.

We now show that the bifurcation equation $b(\lambda, \alpha) = 0$ has the solution $(\lambda, 0)$ for all λ sufficiently near λ_o . To see this, consider the restricted operator $F : \mathbb{R} \times X_{\text{sym}} \rightarrow Y_{\text{sym}}$, which has the zero $(\lambda_o, u_o) \in \mathbb{R} \times X_{\text{sym}}$. The Fréchet derivative at this point is given by the restriction $L|_{X_{\text{sym}}} : X_{\text{sym}} \rightarrow Y_{\text{sym}}$, and in view of $\varphi_o \notin X_{\text{sym}}$ and the fact that it is Fredholm with index zero it also has a continuous inverse. Thus, after possibly reducing the size of Λ_0 , one can apply the implicit function theorem to obtain a smooth function $p : \Lambda_0 \rightarrow X_{\text{sym}}$ which satisfies $F(\lambda, p(\lambda)) = 0$ for all $\lambda \in \Lambda_0$. For each fixed λ , the value $W(\lambda, 0) \in \tilde{X}_0$ is uniquely determined by (14). Since $p(\lambda) \in X_{\text{sym}} \subset \tilde{X}$, this immediately yields that $W(\lambda, 0) = p(\lambda)$ and $b(\lambda, 0) = \psi_o^* P F(\lambda, p(\lambda)) = 0$ for all $\lambda \in \Lambda_0$, again after potentially reducing the size of the open set $\Lambda_0 \ni \lambda_o$.

The trivial solution immediately furnishes the identity $\partial b / \partial \lambda^k(\lambda, 0) = 0$ for all k , and as in the proof of [30, Theorem 3.2] one can verify $b_\alpha(\lambda_o, 0) = 0$. Consider now the function

$$r(\lambda, \alpha) = \begin{cases} \frac{b(\lambda, \alpha)}{\alpha} & \text{for } \alpha \neq 0, \\ b_\alpha(\lambda, 0) & \text{for } \alpha = 0, \end{cases}$$

which is defined and smooth in a neighborhood of $(\lambda_o, 0)$, and which will be used to construct the bifurcating branch which breaks the X_{sym} -symmetry. As in [22, Proposition 2.11] one can show that due to $r(\lambda_o, 0) = b_\alpha(\lambda_o, 0) = 0$ the function r has the expansion

$$\begin{aligned} r(\lambda, \alpha) &= (\lambda - \lambda_o) b_{\lambda\alpha}(\lambda_o, 0) + \frac{1}{2} \alpha b_{\alpha\alpha}(\lambda_o, 0) + \frac{1}{2} (\lambda - \lambda_o)^2 b_{\lambda\lambda\alpha}(\lambda_o, 0) + \\ &\quad + \frac{1}{2} (\lambda - \lambda_o) \alpha b_{\lambda\alpha\alpha}(\lambda_o, 0) + \frac{1}{6} \alpha^2 b_{\alpha\alpha\alpha}(\lambda_o, 0) + R(\lambda - \lambda_o, \alpha) \end{aligned}$$

with $R(\nu, \alpha) = O(\|(\nu, \alpha)\|^3)$. Note that since we do not assume any even or odd symmetry, we have the additional $b_{\alpha\alpha}$ term compared to [22]. Now let $\xi_o \in X_{\text{sym}}$ be defined by the equation $L\xi_o + (I - P)D_\lambda F(\lambda_o, u_o) = 0$. The value is unique since $L|_{X_{\text{sym}}}$ is continuously invertible. Notice, however, that then ξ_o also satisfies (11), since $u_o \in X_{\text{sym}}$ and $F(\mathbb{R}, X_{\text{sym}}) \subset Y_{\text{sym}}$

yield the inclusion $D_\lambda F(\lambda_o, u_o) \in Y_{\text{sym}} \subset R(L)$, and therefore $PD_\lambda F(\lambda_o, u_o) = 0$. According to [22, Table 1] along with the nondegeneracy condition (12) we have

$$r_\lambda(\lambda_o, 0) = b_{\lambda\alpha}(\lambda_o, 0) = \psi_o^* D_{\lambda u} F(\lambda_o, u_o)[\varphi_o] + \psi_o^* D_{uu} F(\lambda_o, u_o)[\varphi_o, \xi_o] \neq 0. \quad (16)$$

An application of the implicit function theorem to $r(\lambda, \alpha) = 0$ then yields a smooth real-valued function $\alpha \mapsto h(\alpha)$, which is defined on an open interval around 0, satisfies $h(0) = \lambda_o$, and such that in a neighborhood of $(\lambda_o, 0)$ we have $r(\lambda, \alpha) = 0$ if and only if $\lambda = h(\alpha)$. This establishes the second solution branch $\alpha \mapsto \alpha\varphi_o + W(h(\alpha), \alpha\varphi_o)$.

It remains to establish the precise bifurcation type. The bifurcation equation $b(\lambda, \alpha) = 0$ has both a trivial solution indexed by λ , which corresponds to the branch of symmetric solutions, and a secondary branch indexed by α , which consists of solutions breaking the primary symmetry. Thus, the specific bifurcation type is determined by the value of the derivative $h'(0)$. If we have $h'(0) \neq 0$, then the bifurcation is transcritical, and it is of pitchfork type if one has $h'(0) = 0$. In the latter case, it is supercritical for $h''(0) > 0$, subcritical if $h''(0) < 0$, and degenerate for $h''(0) = 0$. In fact, in the final case it might not have the typical pitchfork shape, as the shape is determined by the first nonzero derivative of h at 0. Nevertheless, we still refer to it as a pitchfork bifurcation.

In order to find the first two derivatives of the function h describing the secondary branch we twice differentiate its defining equation, i.e., the identity

$$r(h(\alpha), \alpha) = 0 \quad \text{for all } \alpha \text{ near } 0.$$

This leads to the equation

$$r_\lambda(h(\alpha), \alpha)h'(\alpha) + r_\alpha(h(\alpha), \alpha) = 0,$$

as well as to

$$\begin{aligned} r_\lambda(h(\alpha), \alpha)h''(\alpha) + r_{\lambda\lambda}(h(\alpha), \alpha)h'(\alpha)^2 + r_{\lambda\alpha}(h(\alpha), \alpha)h'(\alpha) + \\ + r_{\alpha\lambda}(h(\alpha), \alpha)h'(\alpha) + r_{\alpha\alpha}(h(\alpha), \alpha) = 0. \end{aligned} \quad (17)$$

Evaluating the first equation at $\alpha = 0$ yields, together with [22, Table 1], the explicit formula

$$h'(0) = -\frac{r_\alpha(\lambda_o, 0)}{r_\lambda(\lambda_o, 0)} = -\frac{\frac{1}{2}\psi_o^* D_{uu} F(\lambda_o, u_o)[\varphi_o, \varphi_o]}{\psi_o^* D_{\lambda u} F(\lambda_o, u_o)[\varphi_o] + \psi_o^* D_{uu} F(\lambda_o, u_o)[\varphi_o, \xi_o]}, \quad (18)$$

which by (16) has a nonzero denominator. This establishes both (a) and (b). For later reference, we also note that if one assumes $h'(0) = 0$ and evaluates (17) at $\alpha = 0$, then [22, Table 1] furnishes

$$h''(0) = -\frac{r_{\alpha\alpha}(\lambda_o, 0)}{r_\lambda(\lambda_o, 0)} = -\frac{\frac{1}{3}(\psi_o^* D_{uuu} F(\lambda_o, u_o)[\varphi_o, \varphi_o, \varphi_o] + 3\psi_o^* D_{uu} F(\lambda_o, u_o)[\varphi_o, \eta_*])}{\psi_o^* D_{\lambda u} F(\lambda_o, u_o)[\varphi_o] + \psi_o^* D_{uu} F(\lambda_o, u_o)[\varphi_o, \xi_o]}, \quad (19)$$

where $\eta_* \in \tilde{X}$ is the unique element satisfying

$$L\eta_* + (I - P)D_{uu} F(\lambda_o, u_o)[\varphi_o, \varphi_o] = 0. \quad (20)$$

This completes the proof of the theorem. $\square$

Note that in contrast to the classical approach to $\mathbb{Z}_2$ -symmetry breaking, as well as to our previous results in [22, 30], the above theorem does not require knowledge of a full space decomposition of X or Y . Consequently there is also no requirement on L to have all the spaces in the decomposition as invariant subspaces. The *only* requirement is the invariance of the symmetry space coupled with the Fredholm property, and this renders Theorem 2.4 much more applicable. It needs to be mentioned, however, that in specific applications the knowledge of such a space decomposition can be extremely useful — and this will be the case in our main applications in this paper. For example, equivariance in combination with a suitable space decomposition might allow one to simplify the defining equation (20) for η_* by omitting the projector P , as mentioned for example in [22, Remark 2.13].

2.3 Detecting bifurcations via extended systems

While Theorem 2.4 is a typical sufficient result for establishing the existence of certain bifurcation points, its assumptions are not readily verifiable in all situations. For example, if one is interested in studying secondary bifurcations from a nontrivial branch of solutions, one rarely has access to the functions u_o and φ_o in Assumption 2.2, or even the precise parameter value λ_o of the potential bifurcation point.

To address this problem, the current section is devoted to reformulating the sufficient condition of Theorem 2.4 in the form of a zero-finding problem for a suitable extended system. Our approach follows the ideas from [22, 30], but has been extended to the more general setting of this paper.

For this, consider two Banach spaces X and Y , as well as a smooth parameter-dependent operator $F : \mathbb{R} \times X \rightarrow Y$. In addition, let $X_{\text{sym}} \subset X$ and $Y_{\text{sym}} \subset Y$ denote two closed symmetry subspaces such that

$$F(\mathbb{R}, X_{\text{sym}}) \subset Y_{\text{sym}} , \quad (21)$$

and choose a fixed nontrivial element $\ell \in X^*$. Then in the following we consider the *extended system* for F given by

$$\mathcal{F}(\lambda, u, \varphi) = (0, 0, 0) \quad \text{with} \quad \mathcal{F} : \begin{cases} \mathbb{R} \times X_{\text{sym}} \times X \rightarrow \mathbb{R} \times Y_{\text{sym}} \times Y \\ (\lambda, u, \varphi) \mapsto (\ell(\varphi) - 1, F(\lambda, u), D_u F(\lambda, u)[\varphi]) \end{cases} \quad (22)$$

The operator $\mathcal{F}$ is well-defined in view of the inclusion (21). Moreover, its Fréchet derivative in $\mathcal{L}(\mathbb{R} \times X_{\text{sym}} \times X, \mathbb{R} \times Y_{\text{sym}} \times Y)$ satisfies

$$\begin{aligned} D\mathcal{F}(\lambda, u, \varphi)[\tilde{\lambda}, \tilde{u}, \tilde{\varphi}] &= \left(\ell(\tilde{\varphi}) , \tilde{\lambda} D_{\lambda} F(\lambda, u) + D_u F(\lambda, u)[\tilde{u}] , \right. \\ &\quad \left. \tilde{\lambda} D_{\lambda u} F(\lambda, u)[\varphi] + D_{uu} F(\lambda, u)[\varphi, \tilde{u}] + D_u F(\lambda, u)[\tilde{\varphi}] \right) . \end{aligned} \quad (23)$$

One can readily see that if $(\lambda_o, u_o, \varphi_o)$ is a zero of the extended system, i.e., if we have the identity $\mathcal{F}(\lambda_o, u_o, \varphi_o) = 0$, then (λ_o, u_o) is a symmetric zero of F , and φ_o is a kernel function of the Fréchet derivative of F at this zero. As the following result shows, if we add both the nondegeneracy of the zero of the extended system and the condition $\varphi_o \notin X_{\text{sym}}$ to the mix, then we have actually established a symmetry-breaking bifurcation point.

Theorem 2.5 (Symmetry-breaking bifurcations via extended systems). *Let X and Y be two Banach spaces, and suppose that the parameter-dependent operator $F : \mathbb{R} \times X \rightarrow Y$ is smooth. Moreover, let $X_{\text{sym}} \subset X$ and $Y_{\text{sym}} \subset Y$ be closed subspaces with $F(\mathbb{R}, X_{\text{sym}}) \subset Y_{\text{sym}}$. Finally, assume that for every $\lambda \in \mathbb{R}$ and every $u \in X_{\text{sym}}$ the Fréchet derivative $D_u F(\lambda, u) \in \mathcal{L}(X, Y)$ and its restriction $D_u F(\lambda, u)|_{X_{\text{sym}}} \in \mathcal{L}(X_{\text{sym}}, Y_{\text{sym}})$ are Fredholm operators of index zero. Then we have:*

- (a) *Suppose that all the assumptions of Theorem 2.4 hold, and choose the functional $\ell \in X^*$ in such a way that $\ell(\varphi_o) = 1$. Then the Fréchet derivative $D\mathcal{F}(\lambda_o, u_o, \varphi_o)$ of the operator defined in (22) is invertible, i.e., the solution $(\lambda_o, u_o, \varphi_o) \in \mathbb{R} \times X_{\text{sym}} \times X$ of the extended system*

$$\mathcal{F}(\lambda, u, \varphi) = (0, 0, 0) \quad (24)$$

is an isolated nondegenerate zero.

- (b) *Conversely, if there exists an $\ell \in X^*$ and a $\varphi_o \in X \setminus X_{\text{sym}}$ such that $(\lambda_o, u_o, \varphi_o)$ is a zero of the map $\mathcal{F}$, and if the Fréchet derivative $D\mathcal{F}(\lambda_o, u_o, \varphi_o)$ is invertible, then the nonlinear operator F satisfies all the assumptions of Theorem 2.4.*

In other words, the nonlinear problem $F(\lambda, u) = 0$ has a symmetry-breaking transcritical or pitchfork bifurcation point at $(\lambda_o, u_o) \in \mathbb{R} \times X_{\text{sym}}$ if and only if the triple $(\lambda_o, u_o, \varphi_o)$ is a nondegenerate zero of $\mathcal{F}$ with $\varphi_o \notin X_{\text{sym}}$.

Proof. We first establish the validity of (a). The assumptions of Theorem 2.4 immediately imply that $(\lambda_o, u_o, \varphi_o) \in \mathbb{R} \times X_{\text{sym}} \times X$ is a solution of (24), since we assume $\ell(\varphi_o) = 1$.

For the verification of the injectivity of the Fréchet derivative $D\mathcal{F}(\lambda_o, u_o, \varphi_o)$, suppose there exists a solution $(\tilde{\lambda}, \tilde{u}, \tilde{\varphi}) \in \mathbb{R} \times X_{\text{sym}} \times X$ of the equation

$$D\mathcal{F}(\lambda_o, u_o, \varphi_o)[\tilde{\lambda}, \tilde{u}, \tilde{\varphi}] = (0, 0, 0) . \quad (25)$$

We have to show that this implies $(\tilde{\lambda}, \tilde{u}, \tilde{\varphi}) = (0, 0, 0)$. Suppose first that $\tilde{\lambda} \neq 0$. Then the second component in (25) can be rewritten as

$$D_{\lambda} F(\lambda_o, u_o) + L[\tilde{u}/\tilde{\lambda}] = 0 ,$$

and since $\xi_o \in X_{\text{sym}}$ is the unique solution of (11) one further obtains $\xi_o = \tilde{u}/\tilde{\lambda}$. Now the third component of (25) yields

$$D_{\lambda u} F(\lambda_o, u_o)[\varphi_o] + D_{uu} F(\lambda_o, u_o)[\varphi_o, \xi_o] = -L[\tilde{\varphi}/\tilde{\lambda}] \in R(L) ,$$

which contradicts (12). Thus, the identity $\tilde{\lambda} = 0$ has to be satisfied, and the second components of both (25) and (23) imply $L[\tilde{u}] = 0$, i.e., one has $\tilde{u} \in N(L) \cap X_{\text{sym}} = \{0\}$. Substituting the identities $\tilde{\lambda} = 0$ and $\tilde{u} = 0$ into the third component immediately yields $L\tilde{\varphi} = 0$, and therefore $\tilde{\varphi} = \alpha\varphi_o$. Finally, the first component of (25) implies $0 = \ell(\alpha\varphi_o) = \alpha$, which in turn gives the equality $\tilde{\varphi} = 0$ and completes the verification that $D\mathcal{F}(\lambda_o, u_o, \varphi_o)$ is one-to-one.

Next we consider the surjectivity of $D\mathcal{F}(\lambda_o, u_o, \varphi_o)$. Let $(\tau, y, z) \in \mathbb{R} \times Y_{\text{sym}} \times Y$ be arbitrary. Note first that since $N(L) \cap X_{\text{sym}} = \{0\}$, and thus $y \in Y_{\text{sym}} = L(X_{\text{sym}})$, there

exists a unique $\tilde{u} \in X_{\text{sym}}$ with $L[\tilde{u}] = y$. Due to Assumption 2.2 we have $R(L) = N(\psi_\circ^*)$. Now define

$$\tilde{\lambda} = \frac{\psi_\circ^*(z - D_{uu}F(\lambda_\circ, u_\circ)[\varphi_\circ, \tilde{u}])}{\psi_\circ^*(D_{\lambda u}F(\lambda_\circ, u_\circ)[\varphi_\circ] + D_{uu}F(\lambda_\circ, u_\circ)[\varphi_\circ, \xi_\circ])} ,$$

where the denominator is nonzero in view of (12). This identity can be rewritten in the form

$$\psi_\circ^* \left(\tilde{\lambda} D_{\lambda u}F(\lambda_\circ, u_\circ)[\varphi_\circ] + D_{uu}F(\lambda_\circ, u_\circ)[\varphi_\circ, \tilde{u} + \tilde{\lambda} \xi_\circ] - z \right) = 0 .$$

The choice of $\psi_\circ^*$ implies that the argument in this equation has to be contained in $R(L)$, and since $L[\beta \varphi_\circ] = 0$ for any $\beta \in \mathbb{R}$, there exists a $\tilde{\varphi} \in X$ such that for every β the equation

$$\tilde{\lambda} D_{\lambda u}F(\lambda_\circ, u_\circ)[\varphi_\circ] + D_{uu}F(\lambda_\circ, u_\circ)[\varphi_\circ, \tilde{u} + \tilde{\lambda} \xi_\circ] + L[\tilde{\varphi} + \beta \varphi_\circ] = z \quad (26)$$

holds. Moreover, the definition of $\xi_\circ$ in (11) furnishes

$$\tilde{\lambda} D_\lambda F(\lambda_\circ, u_\circ) + L[\tilde{u} + \tilde{\lambda} \xi_\circ] = \tilde{\lambda} (D_\lambda F(\lambda_\circ, u_\circ) + L[\xi_\circ]) + L[\tilde{u}] = L[\tilde{u}] = y . \quad (27)$$

Finally, the inclusion $\xi_\circ \in X_{\text{sym}}$ implies $\tilde{u} + \tilde{\lambda} \xi_\circ \in X_{\text{sym}}$, and so for all $\beta \in \mathbb{R}$ the identities in (26) and (27) give the second and third components of the desired equation

$$D\mathcal{F}(\lambda_\circ, u_\circ, \varphi_\circ)[\tilde{\lambda}, \tilde{u} + \tilde{\lambda} \xi_\circ, \tilde{\varphi} + \beta \varphi_\circ] = (\tau, y, z) .$$

As the last step, one needs to choose β so that the first component of the equation also holds. Due to $\ell(\varphi_\circ) = 1$, this reduces to $\tau = \ell(\tilde{\varphi} + \beta \varphi_\circ) = \ell(\tilde{\varphi}) + \beta$, and thus $\beta = \tau - \ell(\tilde{\varphi})$. This shows that the Fréchet derivative $D\mathcal{F}(\lambda_\circ, u_\circ, \varphi_\circ)$ is onto, and completes the proof of (a).

We now proceed to the verification of part (b), i.e., we assume that there exists an $\ell \in X^*$ and a $\varphi_\circ \in X \setminus X_{\text{sym}}$ such that $(\lambda_\circ, u_\circ, \varphi_\circ)$ is a zero of $\mathcal{F}$ with invertible Fréchet derivative $D\mathcal{F}(\lambda_\circ, u_\circ, \varphi_\circ) \in \mathcal{L}(\mathbb{R} \times X_{\text{sym}} \times X, \mathbb{R} \times Y_{\text{sym}} \times Y)$. We have to show that then all assumptions of Theorem 2.4 are satisfied.

In view of our assumptions for Theorem 2.5, only a few parts of Assumptions 2.2 and 2.3 have to be established. From $\mathcal{F}(\lambda_\circ, u_\circ, \varphi_\circ) = 0$ one readily obtains both $F(\lambda_\circ, u_\circ) = 0$ and the identities $L[\varphi_\circ] = D_u F(\lambda_\circ, u_\circ)[\varphi_\circ] = 0$. Moreover, the kernel element $\varphi_\circ$ is nontrivial due to $\ell(\varphi_\circ) = 1$, and this yields $\dim N(L) \geq 1$. That this last inequality is in fact an equality follows as in the proof of [22, Theorem 2.12] from the invertibility of $D\mathcal{F}(\lambda_\circ, u_\circ, \varphi_\circ)$.

We still have to verify the nondegeneracy condition (12). For this, let $z \in Y \setminus R(L)$ be arbitrary. Then the invertibility of $D\mathcal{F}(\lambda_\circ, u_\circ, \varphi_\circ)$ guarantees a triple $(\tilde{\lambda}, \tilde{u}, \tilde{\varphi}) \in \mathbb{R} \times X_{\text{sym}} \times X$ which satisfies the identity

$$D\mathcal{F}(\lambda_\circ, u_\circ, \varphi_\circ)[\tilde{\lambda}, \tilde{u}, \tilde{\varphi}] = (0, 0, z) . \quad (28)$$

Let $\xi_\circ$ denote the unique solution of (11). If we assume $\tilde{\lambda} = 0$, then the explicit form of the second component of the above identity yields $0 = \tilde{\lambda} D_\lambda F(\lambda_\circ, u_\circ) + L\tilde{u} = L\tilde{u}$, which in turn implies $\tilde{u} \in N(L) \cap X_{\text{sym}} = \{0\}$. Now the last component of (28) reads

$$z = \tilde{\lambda} D_{\lambda u}F(\lambda_\circ, u_\circ)[\varphi_\circ] + D_{uu}F(\lambda_\circ, u_\circ)[\varphi_\circ, \tilde{u}] + L\tilde{\varphi} = L\tilde{\varphi} ,$$

which contradicts $z \notin R(L)$. Thus, we have to have $\tilde{\lambda} \neq 0$. But then the second component of (28) furnishes $D_\lambda F(\lambda_o, u_o) + L[\tilde{u}/\tilde{\lambda}] = 0$, and this in turn implies $\xi_o = \tilde{u}/\tilde{\lambda}$. Substituting this into the third component of (28), one finally obtains

$$D_{\lambda u} F(\lambda_o, u_o)[\varphi_o] + D_{uu} F(\lambda_o, u_o)[\varphi_o, \xi_o] = (z - L\tilde{\varphi})/\tilde{\lambda} \notin R(L),$$

since we assumed that $z \notin R(L)$, yet clearly $L\tilde{\varphi} \in R(L)$. This establishes the nondegeneracy condition (12) and completes the proof of the theorem. $\square$

As in our previous work [22, 30], the above result enables one to use standard computer-assisted proof techniques to establish the existence of transcritical or pitchfork bifurcations. All one has to do is to identify the correct symmetry space X_{sym} , and find a verifiable condition to make sure that the kernel function φ_o does not lie in this symmetry subspace. This is a significant simplification from our previous approach. However, establishing the actual type of the bifurcation still relies on knowledge of the second derivative $D_{uu} F(\lambda_o, u_o)[\varphi_o, \varphi_o]$. For this, it is extremely useful to have equivariance properties available, since such properties often simplify the task.

2.4 Verifying the symmetry assumptions

We close this section with a few comments concerning Assumption 2.3, which is the main symmetry assumption underlying our above results. In contrast to [22, 30], its formulation is based on the Fredholm property of the linearized operator on symmetry subspaces, rather than using explicit equivariance operators. While at first glance this approach seems to be unnecessarily indirect, it has inherent advantages.

To demonstrate these advantages, consider a partial differential equation problem on the one-dimensional base domain $\Omega = (0, 1)$, such as for example the diblock copolymer equation. In all of the cases studied in [22, 30], the function u_o has reflection symmetries with respect to several points in the domain. These imply that there exists an $n \in \mathbb{N}$ with $n \geq 2$ such that $u_o : \Omega \rightarrow \mathbb{R}$ can be recovered from its restriction to the subdomain $\Omega_{\text{sym}} = (0, 1/n)$ via successive even reflections across the boundary. The differential operators in the diblock copolymer model are based on the Laplacian, so they are automatically equivariant with respect to these reflections. This means that the symmetry spaces X_{sym} and Y_{sym} can be defined as the closed subspaces of X and Y that have these even reflection symmetries around the reflection points $x = k/n$ for $k = 1, \dots, n-1$.

Note that in view of this framework, the symmetry spaces are isomorphic in a straightforward way to the original function spaces for the problem, but this time restricted to the symmetry fundamental domain Ω_{sym} . This is due to the fact that the original problem required homogeneous Neumann boundary conditions, and on the restricted domain the function u_o satisfies the same constraints because of the even reflections mentioned above.

These observations lead to the following approach for establishing the Fredholm property of Assumption 2.3. In many applications, the linearization of the underlying nonlinear operator is a compact perturbation of a leading order differential operator which is based on the Laplacian. Thus, the term consisting of the highest-order derivatives is selfadjoint in the appropriate function spaces, and one can then use the closed range theorem to establish that

it is Fredholm of index zero. The full linearization is a compact perturbation, since usually one adds only lower-order derivatives, or products of the argument functions with coefficient functions in the symmetry subspace. These additional terms preserve the Fredholm property and the index. Most importantly, the same argument works for both the spaces defined over the domain Ω and over the domain Ω_{sym} , and therefore the Fredholm assumptions are immediate consequences both for L and for the restriction $L|_{X_{\text{sym}}}$.

We would like to emphasize that this approach is not restricted to one-dimensional base domains, and can equally be applied in the higher-dimensional setting. For example, consider the square domain $\Omega = (0, 1)^2$ and assume that u_o is a function that has the same even reflection symmetries as $\cos(2\pi x) \cos(3\pi y)$. Then u_o is uniquely determined by its restriction to the fundamental domain $\Omega_{\text{sym}} = (0, 1/2) \times (0, 1/3)$, and the above approach can be applied without change.

Finally, note that establishing the index of the involved Fredholm operators usually can be achieved by relying on the closed range theorem. For more details, we refer the reader to [17, Section III.1]. As we will see later in this paper, the closed range theorem is also important for the discussion of transcritical bifurcation points, since this requires an explicit representation of the range of the linearization L . For this, one can combine orthogonality with a kernel function of the adjoint operator.

3 Results

In this section, we apply our results to two examples. These examples serve to illustrate how the general approach presented in Section 2 can be used to establish the occurrence of symmetry-breaking bifurcations. Some numerically obtained bifurcation diagrams of steady states were already shown in Section 1, for the Ohta–Kawasaki equation in two dimensions (Figure 1) and for the SKT system in one dimension (Figure 2). We also provide here a numerical bifurcation diagram for the Ohta–Kawasaki equation in one dimension (Figure 5), which differs from [30] because we focus on the non-zero mass case, and for the SKT system in two dimensions (Figure 6). We now rigorously establish some of the bifurcations suggested by these numerical simulations.

Theorem 3.1. *Consider the Ohta–Kawasaki equation (1), with $\mu = -0.1$ and $\sigma = 1$. Each line of Table 1 specifies a domain Ω and a value $\bar{\lambda}$ of the bifurcation parameter λ near which a symmetry-breaking bifurcation of steady-states occurs, of the type specified in the third column of the table. The corresponding approximate steady state $\bar{u}$ and approximate kernel function $\bar{\varphi}$ are represented in the figure indicated in the table. For each line, the exact parameter λ_o at which the bifurcation occurs, and the exact steady state u_o and kernel function φ_o satisfy*

$$|\lambda_o - \bar{\lambda}| \leq 10^{-8}, \quad \sup_{x \in \Omega} |u_o(x) - \bar{u}(x)| \leq 10^{-7}, \quad \sup_{x \in \Omega} |\varphi_o(x) - \bar{\varphi}(x)| \leq 10^{-6}.$$

The approximate steady states $\bar{u}$ and approximate kernel functions $\bar{\varphi}$ are given by trigonometric polynomials, whose coefficients are available at [5].

Theorem 3.2. *Consider the SKT system (2), with $d_1 = d_2 = \lambda$ and all other parameters as in (3). Each line of Table 2 specifies a domain Ω and a value $\bar{\lambda}$ of the bifurcation parameter λ*

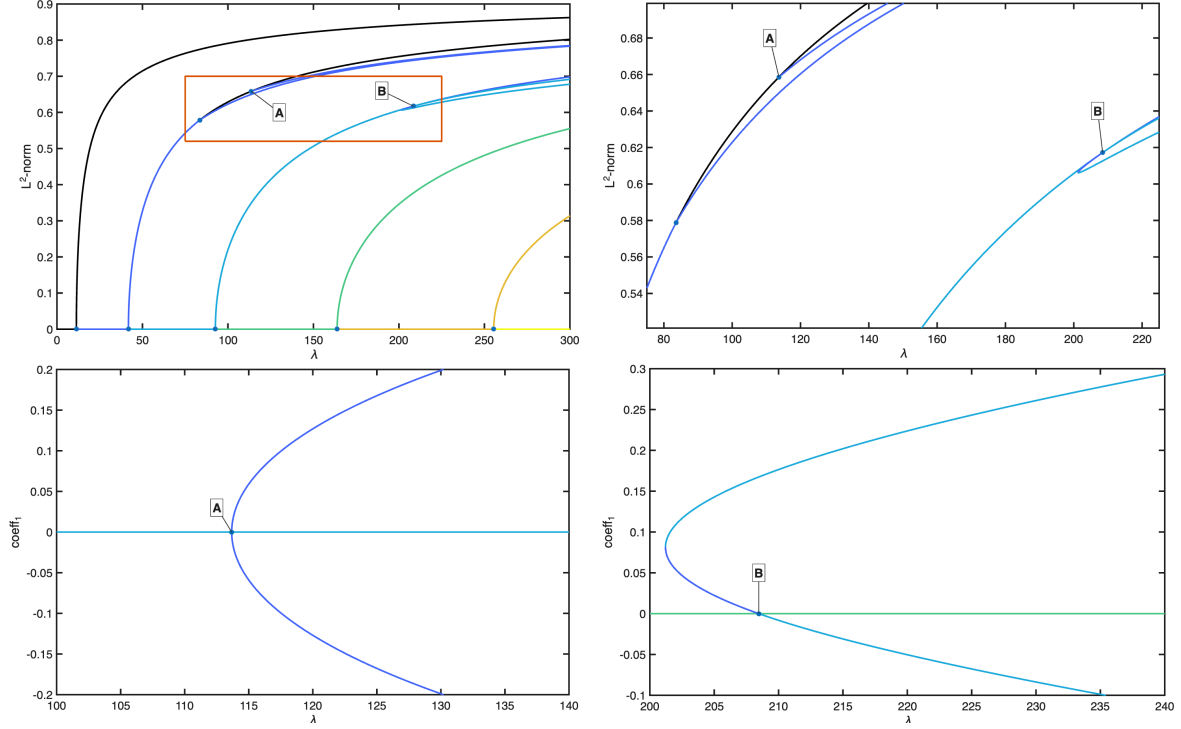


Figure 5: A portion of the bifurcation diagram for the Ohta–Kawasaki equation in one space dimension. The two panels in the top row show a plot of λ versus the L^2 -norm of the solution in the large and in close up. The blue dots represent potential pitchfork and transcritical bifurcations which have been detected using the AUTO software package. The ones labeled A and B are validated in Theorem 3.1. The bottom row contains closeups of two specific bifurcation points indicated in the closeup. In the latter images, the solution measure used for the vertical axis is the first cosine Fourier coefficient u_1 . Plotted in this way, the two bifurcation points have the typical look of a pitchfork and transcritical bifurcation, respectively.

near which a symmetry-breaking bifurcation of steady-states occurs, of the type specified in the third column of the table. The corresponding approximate steady state $\bar{u}$ and approximate kernel function $\bar{\varphi}$ are represented in the figure indicated in the table. For each line, the exact parameter λ_o at which the bifurcation occurs, and the exact steady state u_o and kernel function φ_o satisfy

$$|\lambda_o - \bar{\lambda}| \leq 10^{-14}, \quad \max_{i=1,2} \sup_{x \in \Omega} |(u_o)_i(x) - \bar{u}_i(x)| \leq 10^{-11}, \quad \max_{i=1,2} \sup_{x \in \Omega} |(\varphi_o)_i(x) - \bar{\varphi}_i(x)| \leq 10^{-9}.$$

Here as well, the approximate steady states $\bar{u}$ and approximate kernel functions $\bar{\varphi}$ are given by trigonometric polynomials, whose coefficients are available at [5].

The proofs of Theorem 3.1 and Theorem 3.2 consist in showing that the extended system $\mathcal{F}$ introduced in Section 2 does have a non-degenerate zero $(\lambda_o, u_o, \varphi_o)$ near each of the

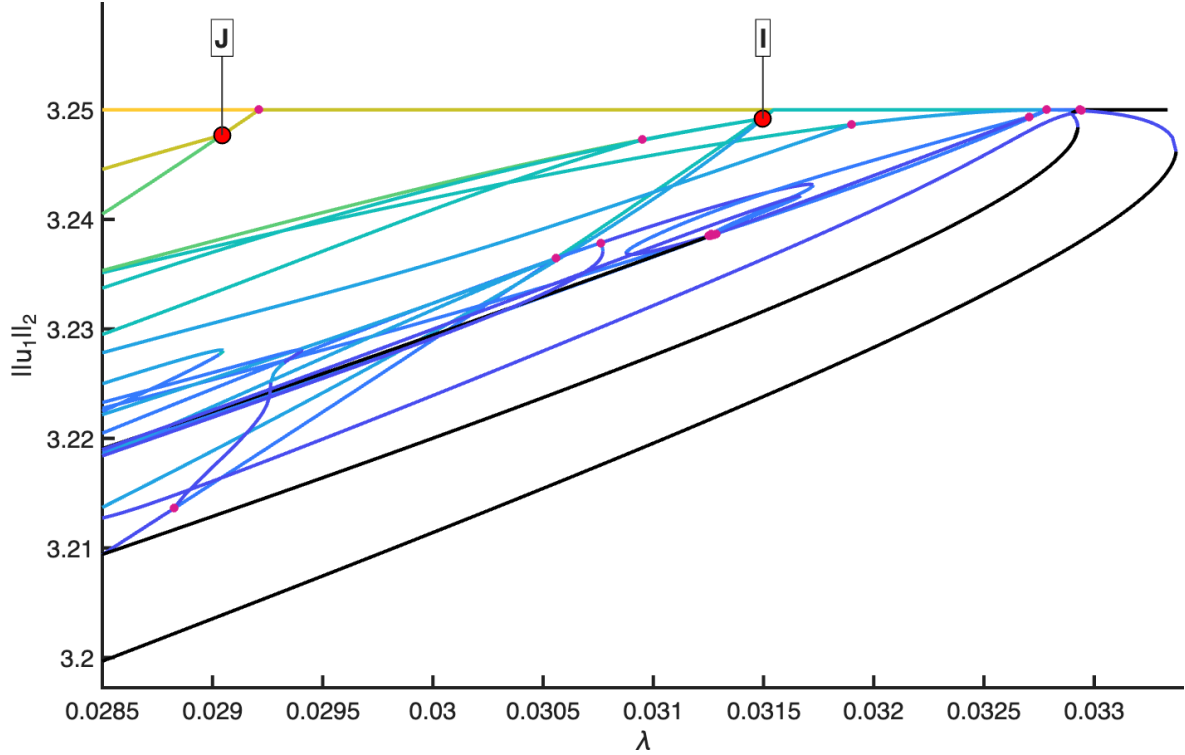


Figure 6: A portion of the bifurcation diagram of steady states for the two-dimensional SKT system (2) on $\Omega = (0, 1) \times (0, 4)$, with parameters as in (3). To be consistent with previous work [19], the L^2 -norm shown is with respect to the first component u_1 only. The dots represent potential pitchfork and transcritical bifurcations which have been detected using the AUTO software package. The ones labeled I and J are validated in Theorem 3.2.

Ω	$\bar{\lambda}$	Bifurcation	Solution	Location	n^{sym}
$(0, 1)$	113.670521515	pitchfork	Fig. 7a	Fig. 5, A	2
$(0, 1)$	208.461521833	transcritical	Fig. 7b	Fig. 5, B	3
$(0, 1) \times (0, 0.97)$	119.663923396	transcritical	Fig. 3	Fig. 1, C	$(1, 3)$
$(0, 1) \times (0, 0.97)$	106.018283308	pitchfork	Fig. 8	Fig. 1, D	$(2, 1)$

Table 1: Data for some symmetry-breaking bifurcations of steady states in the Ohta–Kawasaki equation (1), validated in Theorem 3.1. In the table, Ω is the spatial domain on which the equation is considered, and $\bar{\lambda}$ the approximate value at which the bifurcation occurs. The corresponding approximate steady state and symmetry-breaking eigenfunction are depicted in the figures indicated in the solution column in the table, and the bifurcation location is specified in the following column. Finally, n^{sym} specifies the symmetry space X_{sym} used for the proof, as defined in Section 5.

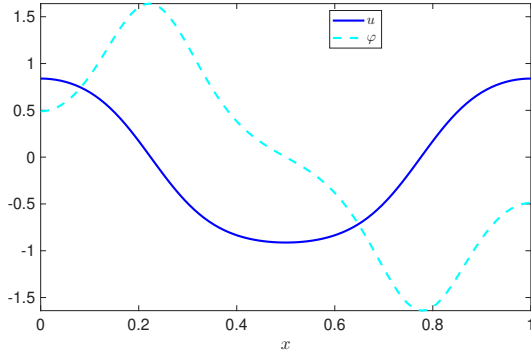
Ω	$\bar{\lambda}$	Bifurcation	Solution	Location	n^{sym}
$(0, 1)$	0.008966808767687	transcritical	Fig. 9a	Fig. 2, E	3
$(0, 1)$	0.013993670967280	pitchfork	Fig. 9b	Fig. 2, F	2
$(0, 1)$	0.003226739574507	pitchfork	Fig. 4	Fig. 2, G	5
$(0, 1)$	0.004452465315501	transcritical	Fig. 9c	Fig. 2, H	3
$(0, 1) \times (0, 4)$	0.031497237998270	transcritical	Fig. 10	Fig. 6, I	$(1, 3)$
$(0, 1) \times (0, 4)$	0.029043593993081	pitchfork	Fig. 11	Fig. 6, J	$(1, 4)$

Table 2: Data for some symmetry-breaking bifurcations of steady states in the SKT system (2), validated in Theorem 3.2. The spatial domain underlying the equation is denoted by Ω , and $\bar{\lambda}$ is the approximate value at which the bifurcation occurs. The corresponding approximate steady state and symmetry-breaking eigenfunction are depicted in the figures indicated in the solution column in the table, and the bifurcation location is specified in the following column. Finally, n^{sym} specifies the symmetry space X_{sym} used for the proof, as defined in Section 5.

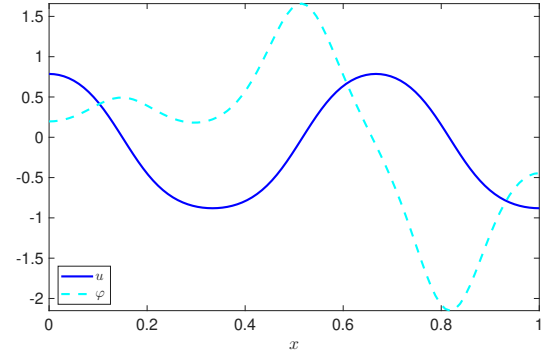
approximate zeros $(\bar{\lambda}, \bar{u}, \bar{\varphi})$ described in Tables 1 and 2 and in the corresponding figures. Theorem 2.5(b) and Theorem 2.4 then show that a symmetry-breaking bifurcation does occur at $(\lambda_o, u_o, \varphi_o)$. The existence of these zeros is obtained using computer-assisted proofs based on a Newton-Kantorovich argument, which are described in Section 4, and which automatically yield that $D\mathcal{F}(\lambda_o, u_o, \varphi_o)$ is indeed invertible. The type of bifurcation is then determined by checking whether assumption (a) or (b) of Theorem 2.4 holds. In the pitchfork case, this is obtained by studying more carefully the symmetries involved (which we note was not required to prove that a symmetry-breaking bifurcation occurs), as explained in Section 5.1. In the transcritical case, we conduct a second computer-assisted proof, this time to rigorously enclose the left eigenfunction ψ_o^* which characterizes the range of L as in (9), which then enables us to check that we are in case (a). This step is described in Section 5.2. All the computer-assisted parts of the proofs can be reproduced using the Matlab code available at [5], together with Intlab [31] for interval arithmetic calculations. Each proof takes between 1 and 20 seconds for the one-dimensional examples, and between 1 and 2 minutes for the two-dimensional examples, on a laptop with an Intel Core Ultra 5 335 processor and 32GB of RAM, but we note that the code could be heavily optimized, especially in the two-dimensional case, for instance by taking advantage of the symmetries as in [42] or [12]. We also emphasize that, for most examples, the obtained error bounds are actually smaller than those reported in Theorem 3.1 and Theorem 3.2, where we only gave the worst case.

4 Computer-assisted proof techniques

One of the main features of Theorem 2.5(b) is that it reformulates the study of symmetry-breaking bifurcations into a problem which can be settled using computer-assisted proofs. In



(a) Label A on Line 1 of Table 1.



(b) Label B on Line 2 of Table 1.

Figure 7: Two different approximate steady states $\bar{u}$ and kernel functions $\bar{\varphi}$ of the Ohta–Kawasaki equation (1) near bifurcation points.

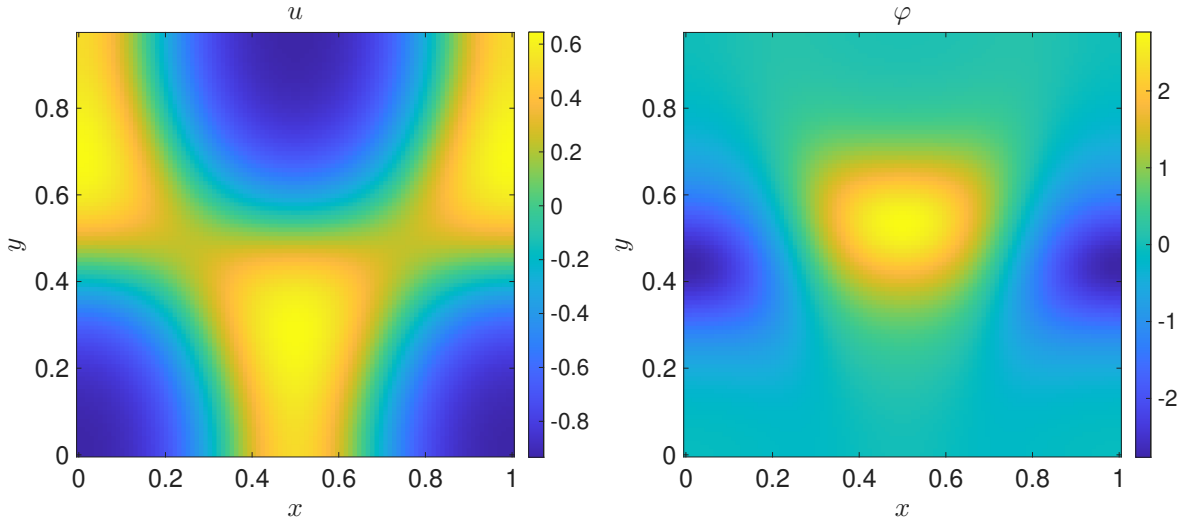
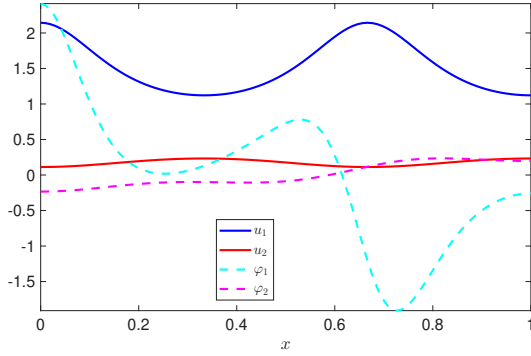


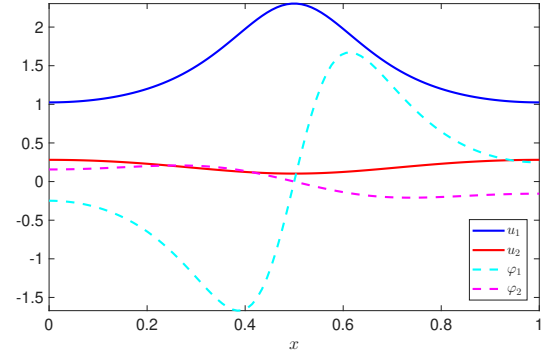
Figure 8: The approximate steady state $\bar{u}$ and kernel function $\bar{\varphi}$ of the Ohta–Kawasaki equation (1) corresponding to label D on Line 4 of Table 1.

this section, we showcase how to apply the result in concrete examples. Our procedure is as follows. We first compute numerically what we hope to be an accurate approximation of a zero of the extended system $\mathcal{F}$ introduced in (22), and then prove a posteriori the existence of a nearby zero of $\mathcal{F}$, using a fixed point argument which also provides us with guaranteed error bounds between the approximate zero and the exact one. The invertibility of the Fréchet derivative required in Theorem 2.5(b) is obtained automatically with our approach. This type of strategy goes back at least to the proof of the Feigenbaum conjecture [20], and is by now common in computer-assisted proofs for differential equations and other dynamical systems [11, 25, 27, 28, 38].

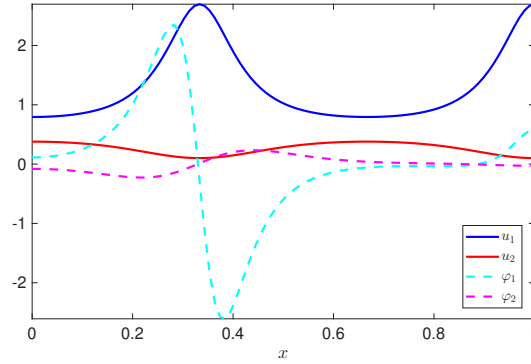
We recall some of the basic mathematical tools underlying such computer-assisted proofs of zeros of systems in Section 4.1. We then consider two specific examples, the Ohta–Kawasaki



(a) Label E on Line 1 of Table 2.



(b) Label F on Line 2 of Table 2.



(c) Label H on Line 4 of Table 2.

Figure 9: Three different approximate steady states $\bar{u}$ and kernel functions $\bar{\varphi}$ of the SKT system (2) near bifurcation points.

model (1) in Section 4.2 and the SKT system (2) in Section 4.3. A strength of our approach is that it does not require very precise information on the involved underlying symmetries. However, one byproduct of this is that Theorem 2.5(b) by itself does not allow us to determine whether the established bifurcation is of transcritical or pitchfork type. This information can often be recovered after the bifurcation has been validated, by checking whether condition (a) or (b) of Theorem 2.4 holds. If the bifurcation is a pitchfork, this requires a deeper analysis of the symmetries (to be done by hand), presented in Section 5.1. On the other hand, if the bifurcation is transcritical, a subsequent computer-assisted proof can be conducted to check condition (a) of Theorem 2.4, and this is presented in Section 5.2.

4.1 The Newton-Kantorovich approach

In this section, we provide sufficient conditions, that can be checked in practice, and which ensure that a zero of $\mathcal{F}$ exists near an element $\bar{x} \in \mathcal{X}$. A common strategy to accomplish this is to consider a Newton-like fixed point operator

$$T : x \mapsto x - A\mathcal{F}(x), \quad (29)$$

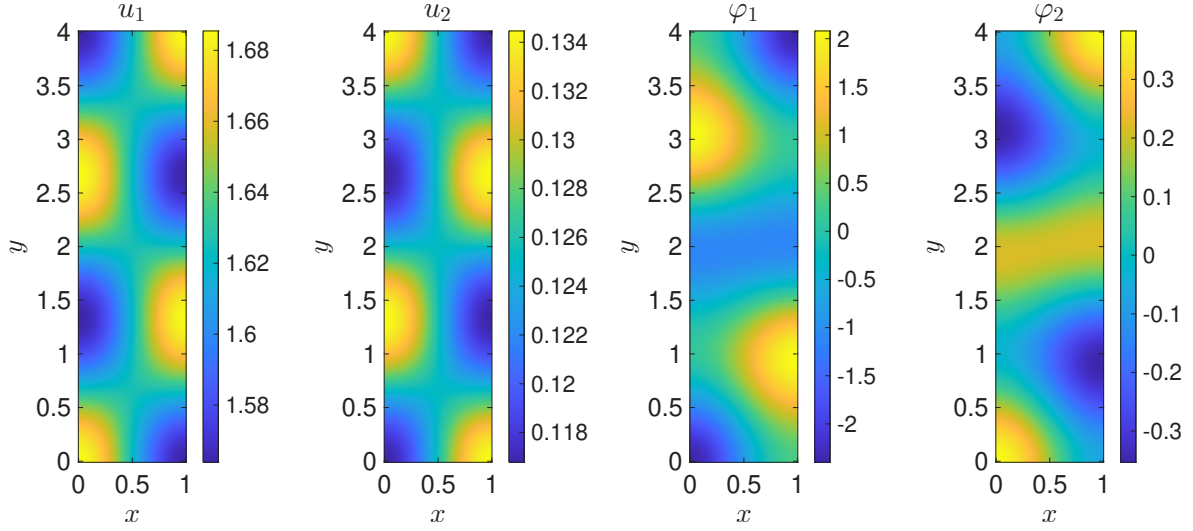


Figure 10: The approximate steady state $\bar{u}$ and kernel function $\bar{\varphi}$ of the SKT system (2) corresponding to label I on Line 5 of Table 2.

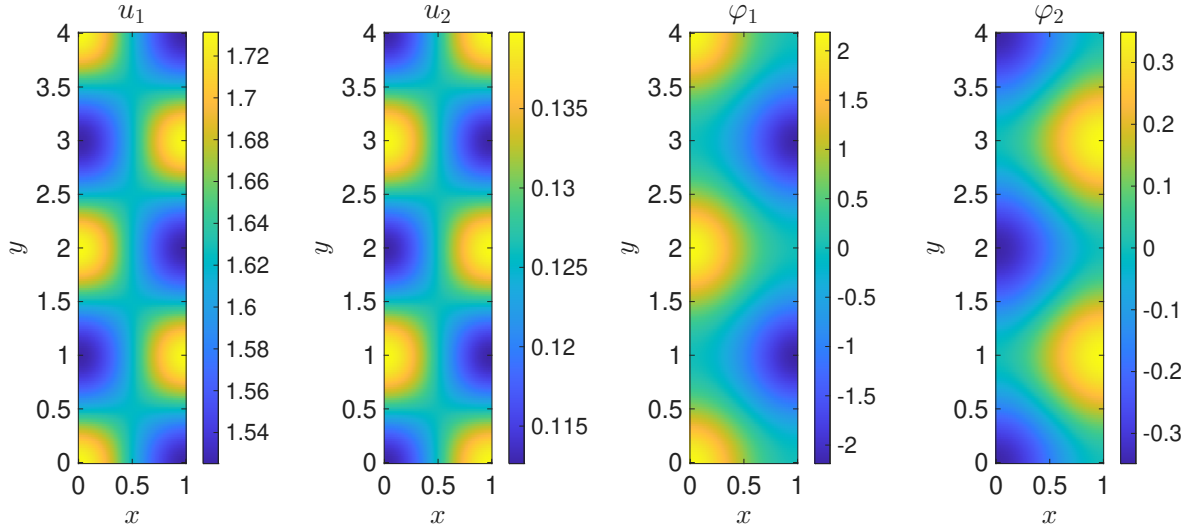


Figure 11: The approximate steady state $\bar{u}$ and kernel function $\bar{\varphi}$ of the SKT system (2) corresponding to label J on Line 6 of Table 2.

where $A : \mathcal{Y} \rightarrow \mathcal{X}$ is an injective linear map approximating $D\mathcal{F}(\bar{x})^{-1}$.

The following statement, based on the Banach fixed point theorem, provides explicit conditions under which T has a unique fixed point near $\bar{x}$. Many similar versions of this theorem have been used in the last decades for computer-assisted proofs. This specific instance is the one introduced in [37, Theorem 2.12], and is well adapted to situations where the space $\mathcal{X}$ has several components, as is the case when we deal with the extended system (22). Its main feature is that the norm on the product space $\mathcal{X}$ is not chosen a priori, and that we do not try

to choose the neighborhood of $\bar{x}$ on which we apply Banach fixed point theorem as a ball for this norm. This provides some extra flexibility when trying to prove that T is a contraction, see [37] for a more detailed discussion.

Theorem 4.1. *Let $M \in \mathbb{N}_{\geq 1}$, $(\mathcal{X}^m, \|\cdot\|_{\mathcal{X}^m})_{m=1}^M$ be Banach spaces, $\mathcal{X} = \Pi_{m=1}^M \mathcal{X}^m$ the product space, and $\pi^m : \mathcal{X} \rightarrow \mathcal{X}^m$ the projections onto the components. Let $r^* = (r_m^*)_{m=1}^M \in \mathbb{R}_{>0}^M$ and $\bar{x} \in \mathcal{X}$. For any $r \in \mathbb{R}_{>0}^M$, we define*

$$\text{Box}(\bar{x}, r) = \{x \in \mathcal{X} : \|\pi^m(x - \bar{x})\|_{\mathcal{X}^m} \leq r_m \text{ for } 1 \leq m \leq M\}.$$

Assume that $T \in C^1(\text{Box}(\bar{x}, r), \mathcal{X})$. For $r, r^ \in \mathbb{R}_{>0}^M$ we say that $r \leq r^*$ if we have $r_m \leq r_m^*$ for all $1 \leq m \leq M$. Finally, we denote partial Fréchet derivatives by D_i .*

Finally, suppose that $Y_m \geq 0$, $Z_{m,i}^1 \geq 0$, and $Z_{m,i,j}^2 \geq 0$ for $1 \leq i, j, m \leq M$ satisfy

$$\|\pi_m(T(\bar{x}) - \bar{x})\|_{\mathcal{X}^m} \leq Y_m, \quad (30)$$

$$\|\pi_m D_i T(\bar{x})\|_{B(\mathcal{X}^i, \mathcal{X}^m)} \leq Z_{m,i}^1, \quad (31)$$

$$\|\pi_m(D_i T(x) - D_i T(\bar{x}))\|_{B(\mathcal{X}^i, \mathcal{X}^m)} \leq \sum_{j=1}^M Z_{m,i,j}^2 \|\pi_j(x - \bar{x})\|_{\mathcal{X}^j} \quad \text{for all } x \in \text{Box}(\bar{x}, r^*). \quad (32)$$

If $r, \eta \in \mathbb{R}_{>0}^M$ with $r \leq r^$ satisfy*

$$Y_m + \sum_{i=1}^M Z_{m,i}^1 r_i + \frac{1}{2} \sum_{i,j=1}^M Z_{m,i,j}^2 r_i r_j \leq r_m \quad (33)$$

$$\sum_{i=1}^M Z_{m,i}^1 \eta_i + \sum_{i,j=1}^M Z_{m,i,j}^2 \eta_i r_j < \eta_m \quad (34)$$

for $1 \leq m \leq M$, then T has a unique fixed point in $\text{Box}(\bar{x}, r)$.

Remark 4.2. *If assumption (34) is satisfied, we can consider on the product space $\mathcal{X}$ the norm given by $\|x\|_{\mathcal{X}} = \max_{1 \leq m \leq M} \frac{1}{\eta_m} \|\pi_m x\|_{\mathcal{X}^m}$, for which we get, for all x in $\text{Box}(\bar{x}, r)$,*

$$\|DT(x)\|_{\mathcal{X}} \leq \max_{1 \leq m \leq M} \frac{1}{\eta_m} \left(\sum_{i=1}^M Z_{m,i}^1 \eta_i + \sum_{i,j=1}^M Z_{m,i,j}^2 \eta_i r_j \right) < 1.$$

Therefore, recalling that $DT(x) = I - ADF(x)$, $ADF(x)$ is invertible. If $D\mathcal{F}(x)$ is a Fredholm operator of index 0, then $D\mathcal{F}(x)$ itself is also invertible.

What we have just shown is that, if the existence of a zero x of $\mathcal{F}$ is obtained via Theorem 4.1, then the invertibility of $D\mathcal{F}(x)$ required in Theorem 2.5(b) automatically holds.

We now consider two examples. For each of them, we first introduce suitable Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ in order to study the $\mathcal{F} = 0$ problem given by the extended system (22), and construct a well chosen approximate inverse A to be used in (29). After that, we derive computable estimates Y_m , $Z_{m,i}^1$ and $Z_{m,i,j}^2$ satisfying assumptions (30)-(32) of Theorem 4.1.

Finally, for each of the approximate bifurcation points obtained numerically and described in Table 1 and Table 2, the code available at [5] can be used to evaluate these estimates and to check that conditions (33)-(34) hold, which proves that a symmetry-breaking bifurcation does occur nearby, according to Theorem 4.1 and Theorem 2.5.

We emphasize that, although we focus on two specific examples below, the entire procedure is rather general, and well understood for a wide class of elliptic problems. The fact that the extended system has several components makes everything more cumbersome to write down, but all the ideas are the same as the ones used for a computer-assisted proof of a regular equilibrium. For this reason, we merely provide the central estimates in the main body of the text, and defer most of the technical details to the Appendix. We also refer readers who are unfamiliar with such computer-assisted proofs to [3, Section 1.3.1] for a basic example, where the same ideas are presented in a simpler context.

4.2 Application to the Ohta–Kawasaki model

In this section, we focus on the Ohta–Kawasaki model (1), which is used to model the evolution of diblock copolymer microstructures [26]. The equilibria of this equation are solutions of the elliptic system given by

$$\begin{cases} F(\lambda, u) = -\Delta (\Delta u + \lambda (u - u^3)) - \lambda \sigma (u - \mu) = 0 & \text{in } \Omega, \\ \partial_n u = \partial_n \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (35)$$

where ∂_n denotes the normal derivative, and we assume that μ and σ are fixed constants. Equilibrium solutions and continuation of solutions with respect to parameters have been studied for this and closely related systems using computer-assisted proofs [6, 29, 33, 40, 41, 42, 43, 44], and in particular [22, 30] with respect to pitchfork bifurcations. However, the methods of the last two papers were restricted in the first case to $\mathbb{Z}_2$ -symmetries, and in the second one to $\mathbb{Z}_n$ -symmetries. In each case, detailed knowledge of the symmetry group X_s was required. Our new theorem expands the cases that are possible to treat, and the current paper is the first one to present transcritical bifurcation points, to consider higher-dimensional domains, and to consider the case of non-zero mass parameter μ .

4.2.1 Sequence space preliminaries

In this work, we consider rectangular domains $\Omega \subset \mathbb{R}^d$, given by $\Omega = \prod_{i=1}^d (0, L_i)$, and study symmetry-breaking bifurcations of steady states of (35). While we only provide examples in dimensions one and two, the theory applies for general d . The shape of the domain together with the Neumann boundary conditions make it natural to look for stationary solutions of (35) as cosine series. For any $s \in \mathbb{R}$, we thus consider the following weighted ℓ^1 spaces:

$$\ell_s^1 := \left\{ u = (u_n)_{n \in \mathbb{N}^d}, \|u\|_{\ell_s^1} := \sum_{n \in \mathbb{N}^d} |u_n| \xi_n^{(s)} < \infty \right\},$$

where

$$\xi_n^{(0)} := \sqrt{(2 - \delta_{n_1, 0}) \dots (2 - \delta_{n_d, 0})}, \quad n \in \mathbb{N}^d, \quad (36)$$

and

$$\xi_n^{(s)} := \xi_n^{(0)}(1 + |n|_2)^s, \quad n \in \mathbb{N}^d, \quad (37)$$

and with $|n|_2 = \sqrt{n_1^2 + \dots + n_d^2}$. To any sequence $u \in \ell_s^1$, $s \geq 0$, we associate the continuous function

$$u : x \mapsto \sum_{n \in \mathbb{N}^d} u_n \xi_n^{(0)} \prod_{i=1}^d \cos\left(\frac{n_i x_i \pi}{L_i}\right). \quad (38)$$

Remark 4.3. *In order to not overburden the notation, we use the same symbol to denote a sequence u in ℓ_s^1 and the function u given by (38), but it should be clear from context which object we are considering in any given instance.*

The weights $\xi_n^{(0)}$ are incorporated into (38) in order to obtain a weight-free formula for the scalar product

$$\langle u, v \rangle_{L^2(\Omega)} := \frac{1}{|\Omega|} \int_{\Omega} u(x)v(x)dx = \sum_{n \in \mathbb{N}^d} u_n v_n. \quad (39)$$

They are also included in the ℓ_s^1 norms so that, for all $s \geq 0$, one has the estimate

$$\sup_{x \in \Omega} |u(x)| \leq \|u\|_{\ell_s^1}.$$

Another important property of the space ℓ_s^1 is that it is a Banach algebra for all $s \geq 0$, for the discrete convolution product corresponding to pointwise products of functions. That is, for $s \geq 0$ and $u, v \in \ell_s^1$, we recall that the product $uv \in \ell_s^1$, where

$$(uv)_n = \xi_n^{(0)} \sum_{k \in \mathbb{Z}^d} \frac{u_{|k|} v_{|n-k|}}{\xi_{|k|}^{(0)} \xi_{|n-k|}^{(0)}}, \quad n \in \mathbb{N}^d, \quad (40)$$

with $|k| = (|k_1|, \dots, |k_d|)$.

Remark 4.4. *The usage of weighted ℓ^1 spaces is common in many other works, and in particular in computer-assisted proofs using cosine series. Within this literature, it is more usual to consider*

$$u : x \mapsto \sum_{n \in \mathbb{N}^d} u_n \left(\xi_n^{(0)}\right)^2 \prod_{i=1}^d \cos\left(\frac{n_i x_i \pi}{L_i}\right)$$

instead of (38), which has the advantage of yielding a formula without weights for the corresponding discrete convolution product. However, in this work we also make use of the Euclidean structure of L^2 , and prioritized having a clean formula without weights for the scalar product (39) rather than for the discrete convolution, which is why we selected the convention (38).

As a final preliminary step, for any $N \in \mathbb{N}^d$ we introduce the projection operators $\Pi^{\leq N}$ and $\Pi^{> N}$ on ℓ_s^1 , given by

$$(\Pi^{\leq N} u)_n := \begin{cases} u_n & \text{if } n \leq N, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad (\Pi^{> N} u)_n := \begin{cases} 0 & \text{if } n \leq N, \\ u_n & \text{otherwise.} \end{cases}$$

Here, $n \leq N$ means that $n_j \leq N_j$ for all $j \in \{1, \dots, d\}$.

4.2.2 Zero finding problem and approximate inverse

In order to study steady states of (35), and symmetry-breaking bifurcations with respect to the parameter λ , we consider $X = \ell_s^1$ for some $s \geq 0$, $Y = \ell_{s-4}^1$, and $F : \mathbb{R} \times X \rightarrow Y$ given by (35) where the product on ℓ_s^1 is given by (40), and for any u in ℓ_s^1 ,

$$(\Delta u)_n = - \sum_{i=1}^d \left(\frac{n_i \pi}{L_i} \right)^2 u_n, \quad n \in \mathbb{N}^d.$$

Following Section 2.3, we consider $\mathcal{X} = \mathbb{R} \times X_{\text{sym}} \times X$, $\mathcal{Y} = \mathbb{R} \times Y_{\text{sym}} \times Y$, and $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{Y}$ the extended system given by (22). Regarding the linear form $l \in X^*$ involved in $\mathcal{F}$, we pick

$$l : \varphi \mapsto \sum_{n \in \mathbb{N}^d} \bar{\varphi}_n \varphi_n,$$

where $\bar{\varphi}$ is some fixed element in X , typically an approximation of the bifurcating eigenfunction. In particular, it will be convenient to assume that we have $\bar{\varphi} \in \Pi^{\leq N} X$ for some $N \in \mathbb{N}^d$. The precise definition of the symmetry subspaces X_{sym} and Y_{sym} in $\mathcal{X}$ and $\mathcal{Y}$ is not important for the moment, but will be given in Section 5.1. At the moment we only assume that we indeed have the inclusion $F(\mathbb{R}, X_{\text{sym}}) \subset Y_{\text{sym}}$, and that the restriction $D_u F(\lambda, u)|_{X_{\text{sym}}} \in \mathcal{L}(X_{\text{sym}}, Y_{\text{sym}})$ is a Fredholm operator of index zero. Notice that the full operator $D_u F(\lambda, u) \in \mathcal{L}(X, Y)$ is a Fredholm operator of index zero, since it is a relatively compact perturbation of $-\Delta^2$, and in each of our examples the same argument readily applies to the restriction $D_u F(\lambda, u)|_{X_{\text{sym}}}$.

With the notations of Theorem 4.1, the spaces are $\mathcal{X}^1 = \mathbb{R}$, $\mathcal{X}^2 = X_{\text{sym}}$ and $\mathcal{X}^3 = X$. Moreover, denoting $x = (\lambda, u, \varphi)$ a generic element of $\mathcal{X}$, the partial derivatives correspond to $D_1 = D_\lambda$, $D_2 = D_u$ and $D_3 = D_\varphi$. Finally, the projections defined on the ℓ_s^1 spaces extend in a natural way to the product spaces $\mathcal{X}$ or $\mathcal{Y}$. For this, given a triple $(\lambda, u, \varphi) \in \mathcal{X}$, one defines both $\Pi^{\leq N}(\lambda, u, \varphi) := (\lambda, \Pi^{\leq N} u, \Pi^{\leq N} \varphi)$ and $\Pi^{> N}(\lambda, u, \varphi) := (0, \Pi^{> N} u, \Pi^{> N} \varphi)$. In particular, we take the convention that $\Pi^{\leq N} \mathbb{R} := \mathbb{R}$ and $\Pi^{> N} \mathbb{R} := \{0\}$.

Suppose now that we have an approximate zero $\bar{x} = (\bar{\lambda}, \bar{u}, \bar{\varphi}) \in \mathcal{X}$ of $\mathcal{F}$. Our goal is to use Theorem 4.1 with the quasi-Newton operator T defined in (29), and therefore we first need to construct a suitable approximate inverse A of $D\mathcal{F}(\bar{x})$. This construction is standard for such semilinear problems in the computer-assisted proof literature, and is a straightforward generalization of the one that could have been used to study a regular steady state of (35).

In order to understand the rationale behind the construction of A , it can be helpful to look at a block-matrix representation of $D\mathcal{F}(\bar{x})$:

$$D\mathcal{F}(\bar{x}) = \begin{pmatrix} 0 & 0 & \ell \\ D_\lambda F(\bar{\lambda}, \bar{u}) & D_u F(\bar{\lambda}, \bar{u}) & 0 \\ D_{\lambda u} F(\bar{\lambda}, \bar{u}) \bar{\varphi} & D_{uu} F(\bar{\lambda}, \bar{u}) \bar{\varphi} & D_u F(\bar{\lambda}, \bar{u}) \end{pmatrix}. \quad (41)$$

Since $D_u F(\lambda, u)$ is given as the sum of $-\Delta^2$ and relatively compact terms, the leading order terms in $D\mathcal{F}(\bar{x})$ are of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Delta^2 & 0 \\ 0 & 0 & -\Delta^2 \end{pmatrix}.$$

At least for the high frequencies, the inverse of the operator $D\mathcal{F}(\bar{x})$ should therefore be well approximated by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Delta^{-2} & 0 \\ 0 & 0 & -\Delta^{-2} \end{pmatrix}.$$

Regarding the low frequencies, we consider an operator $A^{\leq N}$ between the finite-dimensional spaces $\Pi^{\leq N}\mathcal{Y}$ and $\Pi^{\leq N}\mathcal{X}$, obtained by numerically inverting $\Pi^{\leq N}D\mathcal{F}(\bar{\lambda}, \bar{u}, \bar{\varphi})\Pi^{\leq N}$, and then define the full operator A as follows

$$A := A^{\leq N} - \Delta^{-2}\Pi^{>N},$$

with some slight abuse of notation. More precisely, given $y = (\lambda, u, \varphi) \in \mathcal{Y}$, we define

$$\begin{cases} A\Pi^{\leq N}y := A^{\leq N}\Pi^{\leq N}y \\ A\Pi^{>N}y := (0, -\Delta^{-2}\Pi^{>N}u, -\Delta^{-2}\Pi^{>N}\varphi), \end{cases}$$

where, for any u in ℓ_s^1 ,

$$(\Delta^{-1}u)_n := \begin{cases} 0 & n = 0, \\ -\left(\sum_{i=1}^d \left(\frac{n_i\pi}{L_i}\right)^2\right)^{-1} u_n & n \in \mathbb{N}^d \setminus \{0\}. \end{cases}$$

Using the expression Δ^{-1} is a slight abuse of notation, justified by the fact that Δ^{-1} (or in this case its square $\Delta^{-2} = (\Delta^{-1})^2$) only appears composed with the projection $\Pi^{>N}$ on high frequencies.

In the sequel, we assume that the approximate zero $(\bar{\lambda}, \bar{u}, \bar{\varphi})$ of $\mathcal{F}$ belongs to $\Pi^{\leq N}\mathcal{X}$, for the same truncation parameter N as the one used for constructing A . This choice is made with the purpose of simplifying the exposition of the estimates to come, but we emphasize that selecting different truncation parameters for the approximate solution and for the approximate inverse would make the proof more efficient in practice.

4.2.3 Bounds Y , Z^1 and Z^2 for Theorem 4.1

We now provide computable estimates Y_m , $Z_{m,i}^1$ and $Z_{m,i,j}^2$ satisfying assumptions (30)-(32) of Theorem 4.1, for the operator T given by (29), with the spaces $\mathcal{X}$ and $\mathcal{Y}$, and both the map $\mathcal{F}$ and the linear operator A introduced in Section 4.2.2. For notational convenience, we write $x = (\lambda, u, \varphi)$ and $\bar{x} = (\bar{\lambda}, \bar{u}, \bar{\varphi})$. We again refer readers unfamiliar with computer-assisted proofs to [3, Section 1.3.1], where similar estimates are derived in a simpler context, and with more details.

The Y bounds. Since the nonlinear operator $\mathcal{F}$ is polynomial, the evaluation $\mathcal{F}(\bar{x})$ is finite (note that in fact it belongs to $\Pi^{\leq 3N}\mathcal{X}$), and so is $T(\bar{x}) - \bar{x} = A\mathcal{F}(\bar{x})$. Therefore, obtaining the bounds $Y = (Y_1, Y_2, Y_3)$ only amounts to finite computations, and we can control rounding errors by using interval arithmetic [24, 36].

The Z^1 bounds. In order to obtain the Z^1 estimates, one has to estimate the norm of the different components of $DT(\bar{x}) = I - AD\mathcal{F}(\bar{x})$. Such estimates are standard but somewhat tedious to write out fully because of the different components, and can be found in Appendix A. We only give a brief overview of their derivation here. To find Z^1 , we divide our estimates into an estimate on the finite-dimensional space, and an estimate on the tail such that, for $1 \leq m, i \leq 3$,

$$Z_{m,i}^1 := \max \left(Z_{m,i}^{1,\text{finite}}, Z_{m,i}^{1,\text{tail}} \right).$$

We can obtain all terms $Z_{m,i}^{1,\text{finite}}$ with finite numerical calculations using interval arithmetic, as more precisely described in the Appendix. The terms for the tail involve analytical calculations. Our derivations in the appendix show that one can take the following for the estimates of the tail terms:

$$\begin{aligned} Z_{1,i}^{1,\text{tail}} &= 0 & \text{for } i = 1, 2, 3, \\ Z_{m,1}^{1,\text{tail}} &= 0 & \text{for } m = 1, 2, 3, \end{aligned}$$

$$Z_{2,2}^{1,\text{tail}} = Z_{3,3}^{1,\text{tail}} = |\bar{\lambda}| \left(\frac{\|1 - 3\bar{u}^2\|_X}{\varrho_N} + \frac{|\sigma|}{\varrho_{3N}^2} \right), \quad \text{and} \quad Z_{3,2}^{1,\text{tail}} = 6|\bar{\lambda}| \frac{\|\bar{u}\bar{\varphi}\|_X}{\varrho_N}, \quad Z_{2,3}^{1,\text{tail}} = 0,$$

where ϱ_N is defined in (49).

The Z^2 bounds. These estimates are obtained by computing the second derivative of $\mathcal{F}$ and bounding each of its components using the Banach algebra property of X . The calculation involves many terms, but each one is elementary. A concrete example is given in Appendix A.

4.3 Application to the SKT system

In this section we focus on the SKT system (2), which is used to study the evolution of populations with two competing species [35]. The equilibria are solutions of the elliptic system given by

$$\begin{aligned} F(\lambda, u) &= \begin{pmatrix} \Delta (\lambda u_1 + d_{11}u_1^2 + d_{12}u_1u_2) + (r_1u_1 - a_1u_1^2 - b_1u_1u_2) \\ \Delta (\lambda u_2 + d_{21}u_1u_2 + d_{22}u_2^2) + (r_2u_2 - a_2u_2^2 - b_2u_1u_2) \end{pmatrix} = 0 & \text{in } \Omega, \\ &\partial_n u = 0 & \text{on } \partial\Omega \end{aligned} \tag{42}$$

where $u = (u_1, u_2)$ denotes the densities of the two species. One key feature of this model is the presence of cross-diffusion terms, which can give rise to a repulsive effect leading to pattern formation, i.e., to non-homogeneous steady states. There is an extensive body of work dedicated to establishing the existence of such steady states, and we refer in particular to [23] for a thorough study of the competing effects of diffusion (λ), cross-diffusion (d_{12} and d_{21}) and self-diffusion (d_{11} and d_{22}), which leads to (non-)existence results of non-homogeneous steady states. A computer-assisted approach was also introduced in [2], enabling a more quantitative study of these steady states, especially in parameter regimes where many of them co-exist.

Here, we again consider rectangular domains $\Omega \subset \mathbb{R}^d$, given by $\Omega = \prod_{i=1}^d (0, L_i)$, and study symmetry-breaking bifurcations of steady states of (42). To that end, we apply Theorem 2.5(b) via Theorem 4.1, in a very similar fashion to what we did in Section 4.2 for the Ohta–Kawasaki equation. Therefore, we describe the first steps in a more concise way. The only significant new challenge is the presence of nonlinear terms inside the Laplacian, which requires a more involved approximate inverse A . However, we again emphasize that these difficulties are already present when trying to obtain computer-assisted proofs for regular steady states of (42), and the approach proposed in [2] to handle such problems generalizes in a straightforward way to the extended system (22).

4.3.1 Zero finding problem and approximate inverse

We use again the notation introduced in Section 4.2.1. Since (42) is a two-component system, we consider $X = \ell_s^1 \times \ell_s^1$, also with $s \geq 0$, $Y = \ell_{s-2}^1 \times \ell_{s-2}^1$, and $F : \mathbb{R} \times X \rightarrow Y$ from (42), together with the abbreviation $F = (F_1, F_2)$.

The map $\mathcal{F}$ defining the extended system (22) is considered between the spaces $\mathcal{X}$ and $\mathcal{Y}$, where this time one has $\mathcal{X} = \mathbb{R} \times \ell_s^{1,\text{sym}} \times \ell_s^{1,\text{sym}} \times \ell_s^1 \times \ell_s^1$ and $\mathcal{Y} = \mathbb{R} \times \ell_{s-2}^{1,\text{sym}} \times \ell_{s-2}^{1,\text{sym}} \times \ell_{s-2}^1 \times \ell_{s-2}^1$. Once again, the precise definition of the symmetry subspaces $\ell_s^{1,\text{sym}}$ and $\ell_{s-2}^{1,\text{sym}}$ is not important at this stage, and we merely assume that the inclusion $F(\mathbb{R}, X_{\text{sym}}) \subset Y_{\text{sym}}$ is satisfied, and that the restriction $D_u F(\lambda, u)|_{X_{\text{sym}}} \in \mathcal{L}(X_{\text{sym}}, Y_{\text{sym}})$ is a Fredholm operator of index zero. With the notation of Section 4.1 one then has $\mathcal{X}^1 = \mathbb{R}$, $\mathcal{X}^2 = \mathcal{X}^3 = \ell_s^{1,\text{sym}}$, $\mathcal{X}^4 = \mathcal{X}^5 = \ell_s^1$, and, denoting $x = (\lambda, u, \varphi) = (\lambda, u_1, u_2, \varphi_1, \varphi_2)$ a generic element of $\mathcal{X}$, the partial derivatives correspond to $D_1 = D_\lambda$, $D_2 = D_{u_1}$, $D_3 = D_{u_2}$, $D_4 = D_{\varphi_1}$, and $D_5 = D_{\varphi_2}$.

Because of the nonlinear terms inside the Laplacian operator, the construction of the approximate inverse A is slightly more complicated than in Section 4.2.2, and follows the strategy that was introduced in [2]. We still start by numerically constructing $A^{\leq N}$, which is an approximation of $\Pi^{\leq N} D\mathcal{F}(\bar{x})^{-1} \Pi^{\leq N}$. It will be convenient to split $A^{\leq N}$ as a 5×5 -block matrix $(A_{i,j}^{\leq N})_{1 \leq i,j \leq 5}$, where, with the notation of Theorem 4.1, $A_{i,j}^{\leq N} = \pi_i A^{\leq N} \pi_j$. The main difference will be that, in the tail part of A , we can no longer simply use Δ^{-1} . Let us now introduce

$$\alpha = \begin{pmatrix} \bar{\lambda} + 2d_{11}\bar{u}_1 + d_{12}\bar{u}_2 & d_{12}\bar{u}_1 \\ d_{21}\bar{u}_2 & \bar{\lambda} + d_{21}\bar{u}_1 + 2d_{22}\bar{u}_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} 2d_{11}\bar{\varphi}_1 + d_{12}\bar{\varphi}_2 & d_{12}\bar{\varphi}_1 \\ d_{21}\bar{\varphi}_2 & d_{21}\bar{\varphi}_1 + 2d_{22}\bar{\varphi}_2 \end{pmatrix},$$

which are matrices with entries in ℓ_s^1 appearing naturally in $D\mathcal{F}(\bar{x})$. More precisely,

$$D_u F(\bar{\lambda}, \bar{u})u = \Delta(\alpha u) + \text{lower order terms}, \quad (43)$$

and

$$D_{uu} F(\bar{\lambda}, \bar{u})(u, \bar{\varphi}) = \Delta(\beta u) + \text{lower order terms}.$$

Thus, looking back at (41), the leading order terms in $D\mathcal{F}(\bar{x})$ for this example are

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta(\alpha \cdot) & 0 \\ 0 & \Delta(\beta \cdot) & \Delta(\alpha \cdot) \end{pmatrix}.$$

Next we numerically compute

$$\bar{\omega} = \begin{pmatrix} \bar{\omega}_{11} & \bar{\omega}_{12} \\ \bar{\omega}_{21} & \bar{\omega}_{22} \end{pmatrix} \quad \text{and} \quad \bar{\gamma} = \begin{pmatrix} \bar{\gamma}_{11} & \bar{\gamma}_{12} \\ \bar{\gamma}_{21} & \bar{\gamma}_{22} \end{pmatrix},$$

with entries $\bar{\omega}_{ij}$ and $\bar{\gamma}_{ij}$ in $\Pi^{\leq N} X$, such that $\bar{\omega} \approx \alpha^{-1}$ and $\bar{\gamma} \approx -\bar{\omega}\beta\bar{\omega}$, which then means that

$$\begin{pmatrix} \bar{\omega} & 0 \\ \bar{\gamma} & \bar{\omega} \end{pmatrix} \approx \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix}^{-1}.$$

Finally one defines A by having it act like $\begin{pmatrix} \bar{\omega} & 0 \\ \bar{\gamma} & \bar{\omega} \end{pmatrix} \Delta^{-1}$, except on the finite-dimensional subspaces defined by $\Pi^{\leq N}$, on which we use $A^{\leq N}$. More precisely, for all $y = (\lambda, u, \varphi) \in \mathcal{Y}$,

$$\begin{cases} A\Pi^{\leq N}y := A^{\leq N}\Pi^{\leq N}y + (0, \Pi^{>N}[\bar{\omega}\Delta^{-1}\Pi^{\leq N}u], \Pi^{>N}[\bar{\gamma}\Delta^{-1}\Pi^{\leq N}u + \bar{\omega}\Delta^{-1}\Pi^{\leq N}\varphi]) \\ A\Pi^{>N}y := (0, \bar{\omega}\Delta^{-1}\Pi^{>N}u, \bar{\gamma}\Delta^{-1}\Pi^{>N}u + \bar{\omega}\Delta^{-1}\Pi^{>N}\varphi). \end{cases}$$

We refer to [2] for a more schematic description of A , in a simpler context with fewer components in $\mathcal{F}$. We also note that, as explained in [2], a successful application of Theorem 4.1 to this problem implies that α is invertible, and therefore we see from (43) that $D_u F(\bar{\lambda}, \bar{u})$ is indeed a Fredholm operator of index 0.

4.3.2 Bounds Y , Z^1 and Z^2 for Theorem 4.1

The way the bounds are derived is very similar to what we did in Section 4.2.3, with technical adaptations due to the more complicated structure of A . These bounds are contained in Appendix B.

5 Determining the bifurcation type

So far, we have given a description of how to show that a bifurcation occurs. In this section, we present details on how one can distinguish the bifurcation type.

5.1 Establishing bifurcation symmetries

We now describe more precisely some of the various symmetries that we encountered in our numerical explorations of the bifurcation diagrams for the Ohta–Kawasaki equation (35) and the SKT system (42), for $\Omega = \prod_{i=1}^d (0, L_i)$, in dimensions $d = 1$ and $d = 2$. In particular, we identify some cases in which we can prove that the bifurcation must be of pitchfork type. Using the notations of Section 2 and Theorem 2.4(b), we therefore strive to prove that

$$\psi_{\circ}^*(D_{uu}F(\lambda_{\circ}, u_{\circ})[\varphi_{\circ}, \varphi_{\circ}]) = 0. \quad (44)$$

To that end, the following observation will prove to be helpful.

Lemma 5.1. *Under Assumptions 2.2 and 2.3, $\psi_{\circ}^*|_{Y_{\text{sym}}} = 0$.*

Proof. As already noted in the proof of Theorem 2.4, $L|_{X_{\text{sym}}} : X_{\text{sym}} \rightarrow Y_{\text{sym}}$ is onto. Therefore, $Y_{\text{sym}} \subset R(L) = N(\psi_{\circ}^*)$. $\square$

5.1.1 The one-dimensional case

We first consider both the SKT system (42) and the Ohta–Kawasaki equation (35) on the one-dimensional domain $\Omega = (0, L)$. The symmetries we need to consider can be encoded in symmetry spaces of the form

$$\ell_s^{1,\text{sym}} = \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin n^{\text{sym}}\mathbb{N}\},$$

for some $n^{\text{sym}} \in \mathbb{N}_{\geq 2}$, which is specified in Table 1 and Table 2 for each of our examples.

Our goal here is to prove that, for $n^{\text{sym}} = 2$ and $n^{\text{sym}} = 5$ (possibly under additional assumptions), the symmetry-breaking bifurcations we proved to exist in Section 3 must be of pitchfork type, by showing that $D_{uu}F(\lambda_o, u_o)[\varphi_o, \varphi_o]$ has to lie in a subspace which is contained in the kernel of ψ_o^* . For this, we first introduce

$$\ell_s^{1,\text{asym}} = \{u \in \ell_s^1, u_n = 0 \text{ for all } n \in n^{\text{sym}}\mathbb{N}\},$$

such that $\ell_s^1 = \ell_s^{1,\text{sym}} \oplus \ell_s^{1,\text{asym}}$. We then take $X_{\text{sym}} = \ell_s^{1,\text{sym}}$ and $Y_{\text{sym}} = \ell_{s-4}^{1,\text{sym}}$ for the Ohta–Kawasaki equation, and $X_{\text{sym}} = \ell_s^{1,\text{sym}} \times \ell_s^{1,\text{sym}}$, $Y_{\text{sym}} = \ell_{s-2}^{1,\text{sym}} \times \ell_{s-2}^{1,\text{sym}}$ for the SKT system. Introducing $X_{\text{asym}} = \ell_s^{1,\text{asym}}$ and $Y_{\text{asym}} = \ell_{s-4}^{1,\text{asym}}$, or $X_{\text{asym}} = \ell_s^{1,\text{asym}} \times \ell_s^{1,\text{asym}}$ and $Y_{\text{asym}} = \ell_{s-2}^{1,\text{asym}} \times \ell_{s-2}^{1,\text{asym}}$, the trigonometric formula $\cos(p)\cos(q) = \frac{1}{2}\cos(p+q) + \frac{1}{2}\cos(p-q)$ yields that $L(X_{\text{asym}}) \subset Y_{\text{asym}}$. In particular, Assumption 2.3 implies $\varphi_o \in X_{\text{asym}}$. For the remainder of the discussion, we have to distinguish cases.

Case $n^{\text{sym}} = 2$. For this specific value of n^{sym} , the product of two elements of $\ell_s^{1,\text{asym}}$ is in $\ell_s^{1,\text{sym}}$ (the product of two odd functions is even), therefore $D_{uu}F(\lambda_o, u_o)[\varphi_o, \varphi_o] \in Y_{\text{sym}}$, and by Lemma 5.1 we must have (44). In summary, when $n^{\text{sym}} = 2$ we can only have pitchfork bifurcations.

Case $n^{\text{sym}} = 5$. Here the situation is more subtle, because the product of two elements of $\ell_s^{1,\text{asym}}$ no longer needs to be in $\ell_s^{1,\text{sym}}$. We therefore need to further decompose the space $\ell_s^{1,\text{asym}}$, and introduce

$$\ell_s^{1,\text{asym}_1} = \{u \in \ell_s^{1,\text{asym}}, u_n = 0 \text{ for all } n \in (5\mathbb{N} + 2) \cup (5\mathbb{N} + 3)\},$$

$$\ell_s^{1,\text{asym}_2} = \{u \in \ell_s^{1,\text{asym}}, u_n = 0 \text{ for all } n \in (5\mathbb{N} + 1) \cup (5\mathbb{N} + 4)\},$$

such that $\ell_s^{1,\text{asym}} = \ell_s^{1,\text{asym}_1} \oplus \ell_s^{1,\text{asym}_2}$. The spaces X_{asym_i} and Y_{asym_i} for $i = 1, 2$ are defined accordingly. Since $u_o \in X_{\text{sym}}$, the above-mentioned trigonometric formula for $\cos(p)\cos(q)$ implies the inclusions $L(X_{\text{asym}_1}) \subset Y_{\text{asym}_1}$ and $L(X_{\text{asym}_2}) \subset Y_{\text{asym}_2}$.

We now make two additional assumptions, namely that $\varphi_o \notin X_{\text{asym}_2}$, and that the restriction $L|_{X_{\text{asym}_2}} : X_{\text{asym}_2} \rightarrow Y_{\text{asym}_2}$ is a Fredholm operator of index 0. One then readily obtains $\varphi_o \in X_{\text{asym}_1}$, and the same arguments as in Lemma 5.1 yield that $\psi_o^*|_{Y_{\text{sym}} \oplus Y_{\text{asym}_2}} = 0$. On the other hand, the product of two elements of ℓ_s^{1,asym_1} lies in $\ell_s^{1,\text{sym}} \oplus \ell_s^{1,\text{asym}_2}$ (using again the trigonometric formula for $\cos(p)\cos(q)$), and therefore $D_{uu}F(\lambda_o, u_o)[\varphi_o, \varphi_o]$ belongs to

the space $Y_{\text{sym}} \oplus Y_{\text{asym}_2}$, and one obtains (44). In summary, under the above additional assumptions (which are easy to verify in practice) that $\varphi_\circ \notin X_{\text{asym}_2}$ and that the restriction $L|_{X_{\text{asym}_2}} : X_{\text{asym}_2} \rightarrow Y_{\text{asym}_2}$ is a Fredholm operator of index 0, we have a pitchfork bifurcation for $n^{\text{sym}} = 5$. We note that ℓ_s^{1,asym_1} and ℓ_s^{1,asym_2} play interchangeable roles here, so the same conclusion would hold if $\varphi_\circ \notin X_{\text{asym}_1}$ and $L|_{X_{\text{asym}_1}} : X_{\text{asym}_1} \rightarrow Y_{\text{asym}_1}$ is a Fredholm operator of index 0.

There are other values of n^{sym} , such as for example $n^{\text{sym}} = 3$, for which no similar argument can be made to work. Unsurprisingly, in practice we observe transcritical bifurcations for such values of n^{sym} . A computer-assisted method for proving that a given bifurcation is indeed transcritical is given in Section 5.2.

5.1.2 The two-dimensional case

As could be expected, there is a broader variety of symmetries in the two-dimensional case, i.e., when we consider the SKT system (42) or the Ohta–Kawasaki equation (35) on the domain $\Omega = (0, L_1) \times (0, L_2)$. Below are some examples of subspaces of ℓ_s^1 encoding some of the symmetries that we found numerically (and that were broken through some bifurcation):

$$\begin{aligned} & \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin (n_1^{\text{sym}} 2\mathbb{N} \times n_2^{\text{sym}} 2\mathbb{N}) \cup (n_1^{\text{sym}} (2\mathbb{N} + 1) \times n_2^{\text{sym}} (2\mathbb{N} + 1))\}, \\ & \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin n_1^{\text{sym}} \mathbb{N} \times n_2^{\text{sym}} \mathbb{N}\}, \\ & \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin n_1^{\text{sym}} (2\mathbb{N} + 1) \times n_2^{\text{sym}} (2\mathbb{N} + 1)\}, \\ & \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin n_1^{\text{sym}} \mathbb{N} \times n_2^{\text{sym}} (2\mathbb{N} + 1)\}, \end{aligned}$$

where each time $n^{\text{sym}} \in (\mathbb{N}_{\geq 1})^2$. Each of these was observed in the two-dimensional Ohta–Kawasaki model. We now focus on the first case, namely

$$\ell_s^{1,\text{sym}} = \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin (n_1^{\text{sym}} 2\mathbb{N} \times n_2^{\text{sym}} 2\mathbb{N}) \cup (n_1^{\text{sym}} (2\mathbb{N} + 1) \times n_2^{\text{sym}} (2\mathbb{N} + 1))\},$$

which is the one that also occurs for the two-dimensional SKT system, and show how the bifurcation can be proven to be of pitchfork type for some values of n^{sym} (sometimes under extra assumptions, that are easy to verify in practice). As in Section 5.1.1, one mainly has to introduce a suitable complement $\ell_s^{1,\text{asym}}$ of $\ell_s^{1,\text{sym}}$, given by

$$\ell_s^{1,\text{asym}} = \{u \in \ell_s^1, u_n = 0 \text{ for all } n \in (n_1^{\text{sym}} 2\mathbb{N} \times n_2^{\text{sym}} 2\mathbb{N}) \cup (n_1^{\text{sym}} (2\mathbb{N} + 1) \times n_2^{\text{sym}} (2\mathbb{N} + 1))\},$$

and possibly to split $\ell_s^{1,\text{asym}}$ into further subspaces that are stable under L , and behave appropriately under products. The subspaces X_{sym} , X_{asym} , and Y_{sym} , etc. are defined as in the one-dimensional case.

Case $n^{\text{sym}} = (2, 1)$ for the Ohta–Kawasaki equation. We have

$$\ell_s^{1,\text{sym}} = \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin (4\mathbb{N} \times 2\mathbb{N}) \cup ((4\mathbb{N} + 2) \times (2\mathbb{N} + 1))\},$$

and we consider

$$\ell_s^{1,\text{asym}_1} = \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin (4\mathbb{N} \times (2\mathbb{N} + 1)) \cup ((4\mathbb{N} + 2) \times 2\mathbb{N})\},$$

$$\ell_s^{1,\text{asym}_2} = \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin ((4\mathbb{N} + 1) \times \mathbb{N}) \cup ((4\mathbb{N} + 3) \times \mathbb{N})\},$$

so that

$$\ell_s^1 = \ell_s^{1,\text{sym}} \oplus \ell_s^{1,\text{asym}_1} \oplus \ell_s^{1,\text{asym}_2}.$$

It is straightforward to check that both $L(X_{\text{asym}_1}) \subset Y_{\text{asym}_1}$ and $L(X_{\text{asym}_2}) \subset Y_{\text{asym}_2}$ are satisfied. Assuming that $\varphi_o \notin \ell_s^{1,\text{asym}_2}$ holds, one then obtains $\varphi_o \in \ell_s^{1,\text{asym}_1}$, and straightforward calculations show that $\varphi_o^2 \in \ell_s^{1,\text{sym}}$ and $u_o \varphi_o^2 \in \ell_s^{1,\text{sym}}$. Thus,

$$D_{uu}F(\lambda_o, u_o)[\varphi_o, \varphi_o] = 6\lambda_o \Delta(u_o \varphi_o^2) \in \ell_s^{1,\text{sym}}.$$

Finally, using Lemma 5.1, one indeed obtains the equality in (44). In summary, under the additional assumption $\varphi_o \notin \ell_s^{1,\text{asym}_2}$, the bifurcation is of pitchfork type.

Case $n^{\text{sym}} = (1, 4)$ for the SKT system. The procedure is exactly the same, with

$$\ell_s^{1,\text{sym}} = \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin (2\mathbb{N} \times 8\mathbb{N}) \cup ((2\mathbb{N} + 1) \times (8\mathbb{N} + 4))\},$$

$$\ell_s^{1,\text{asym}_1} = \{u \in \ell_s^1, u_n = 0 \text{ for all } n \notin ((2\mathbb{N} + 1) \times 8\mathbb{N}) \cup (2\mathbb{N} \times (8\mathbb{N} + 4))\},$$

$$\ell_s^{1,\text{asym}_2} = \left\{ u \in \ell_s^1, u_n = 0 \text{ for all } n \notin \mathbb{N} \times \bigcup_{j \notin \{0,4\}} (8\mathbb{N} + j) \right\},$$

and taking into account the different formula for $D_{uu}F(\lambda_o, u_o)[\varphi_o, \varphi_o]$. Under the additional assumption $\varphi_o \notin X_{\text{asym}_2}$, the bifurcation is a pitchfork bifurcation.

There are other values of n^{sym} , such as for example $n^{\text{sym}} = (1, 3)$, for which no similar argument can be made to work. Unsurprisingly, in practice we observe transcritical bifurcations for such values of n^{sym} . A computer-assisted method for proving that a given bifurcation is indeed transcritical is presented below.

5.2 Proving transcritical bifurcations

In this last section, we present a method which allows us to prove that a given bifurcation, obtained via Theorem 4.1 and Theorem 2.5(b), is transcritical. Using the notations of Section 2 and Theorem 2.4(a), we therefore strive to prove that

$$\psi_o^*(D_{uu}F(\lambda_o, u_o)[\varphi_o, \varphi_o]) \neq 0. \quad (45)$$

In contrast to the pitchfork bifurcations discussed in Section 5.1, whose existence follows from specific structural assumptions on the solutions, transcritical bifurcations are generic, which makes it harder to prove by hand that a given bifurcation is actually transcritical. Informally speaking, if $\psi_o^*(D_{uu}F(\lambda_o, u_o)[\varphi_o, \varphi_o])$ is in fact equal to 0, then there should be a good reason for why this is the case, and we might be able to understand it. However, there may not be a clear reason why this quantity is different from 0.

Fortunately, this is one of the many situations where computer-assisted proofs complement more traditional techniques very well. Indeed, once we have rigorous enclosures of λ_o , u_o ,

$\varphi_\circ$, and $\psi_\circ$, we readily get a rigorous enclosure of $\psi_\circ^*(D_{uu}F(\lambda_\circ, u_\circ)[\varphi_\circ, \varphi_\circ])$. From such an enclosure, one cannot hope to prove that (44) holds, but as we saw in Section 5.1 this can be done by hand. On the other hand, provided the enclosure is sharp enough, it can be used to prove (45), which was the case that seemed harder to study by hand.

The computer-assisted proofs for the existence of a bifurcation described in Sections 4.1–4.3 already gave us relatively tight enclosures for $\lambda_\circ$, $\varphi_\circ$, and $u_\circ$. The missing piece is an explicit enclosure of $\psi_\circ^*$, which we are going to obtain using a separate computer-assisted proof. Once this is done, we readily get a computable enclosure for $\psi_\circ^*(D_{uu}F(\lambda_\circ, u_\circ)[\varphi_\circ, \varphi_\circ])$, and can check whether it implies (45).

Recall that we want to find $\psi_\circ^* \in Y^*$ such that $R(L) = N(\psi_\circ^*)$. By the Fredholm alternative, this is equivalent to finding a nontrivial $\psi_\circ^* \in N(L^*)$. The equation $L^*\psi^* = 0$ is the one we are going to solve using a computer-assisted proof which is very similar to the ones described earlier in Section 4.

We now focus on rigorously enclosing ψ^* for the Ohta–Kawasaki equation of Section 4.2, but the procedure we propose is easily adapted to the SKT system of Section 4.3. First, we recall that we work here with spaces X and Y which are weighted ℓ^1 spaces, and whose duals are thus (isomorphic to) weighted ℓ^∞ spaces. We therefore introduce, for $s \in \mathbb{R}$, the spaces

$$\ell_{-s}^\infty := \left\{ u = (u_n)_{n \in \mathbb{N}^d}, \|u\|_{\ell_{-s}^\infty} := \sup_{n \in \mathbb{N}^d} \frac{|u_n|}{\xi_n^{(s)}} < \infty \right\}.$$

We then have that $(\ell_s^1)^* \cong \ell_{-s}^\infty$. In particular, any $\psi^* \in (\ell_s^1)^*$ can be uniquely represented by the element $\psi \in \ell_{-s}^\infty$ such that, for all $u \in \ell_s^1$,

$$\psi^*(u) = \langle \psi^*, u \rangle_{(\ell_s^1)^*, \ell_s^1} = \sum_{n \in \mathbb{N}^d} \psi_n u_n.$$

With this identification, we have $\|\psi^*\|_{(\ell_s^1)^*} = \|\psi\|_{\ell_{-s}^\infty}$.

In Section 4.2, for the purpose of enclosing $\lambda_\circ$, $u_\circ$, and $\varphi_\circ$, we considered the two sequence spaces $X = \ell_s^1$ and $Y = \ell_{s-4}^1$, for some $s \geq 0$. In order to enclose $\psi_\circ$, we thus consider the zero-finding problem $\mathcal{G}^* : \mathbb{R} \times \ell_{-(s-4)}^\infty \rightarrow \mathbb{R} \times \ell_{-s}^\infty$ given by

$$\mathcal{G}^*(\mu, \psi) = (\psi^*(\bar{\varphi}) - 1, D_u F(\lambda_\circ, u_\circ)^* \psi - \mu \psi),$$

where $\bar{\varphi}$ is the approximation of $\varphi_\circ$ used to validate the existence of the bifurcation. At first glance, also solving for μ and not only for ψ might look surprising, as we already know we want μ to be equal to 0. However, if we want to successfully apply Theorem 4.1, the solution we look for must be locally unique, and adding μ as an unknown is a natural way of regularizing the problem. Moreover, this extra unknown is no impediment, as there is a simple a posteriori check that can be used to prove that, if we have a zero of $\mathcal{G}^*$, then μ was in fact equal to 0.

Lemma 5.2. *Let $(\mu_0, \psi_\circ)$ be a zero of the above-defined operator $\mathcal{G}^*$. In addition, suppose that $\|\psi_\circ^* - \bar{\psi}^*\|_{Y^*} \leq r_\psi$ and that $\|\varphi_\circ - \bar{\varphi}\|_X \leq r_\varphi$. If*

$$\|\bar{\psi}^*\|_{Y^*} r_\varphi + \|\bar{\varphi}\|_Y r_\psi + r_\psi r_\varphi < |\bar{\psi}^*(\bar{\varphi})|, \quad (46)$$

then the identity $D_u F(\lambda_\circ, u_\circ)^ \psi_\circ = 0$ is satisfied.*

Proof. Since $(\mu_0, \psi_\circ)$ is a zero of $\mathcal{G}^*$, $\psi_\circ$ is an eigenfunction of $D_u F(\lambda_\circ, u_\circ)^*$, with eigenvalue μ_0 , whereas $\varphi_\circ$ is an eigenfunction of $D_u F(\lambda_\circ, u_\circ)$, with eigenvalue 0. By a classical argument, if μ_0 were to be nonzero, then we would have $\psi_\circ^*(\varphi_\circ) = 0$. Indeed,

$$\mu_0 \psi_\circ^*(\varphi_\circ) = \psi_\circ^*(D_u F(\lambda_\circ, u_\circ) \varphi_\circ) = 0.$$

Yet, our assumption (46) implies that $\psi_\circ^*(\varphi_\circ) \neq 0$, and therefore one must have $\mu_0 = 0$. $\square$

In order to get a rigorous enclosure for a zero of $\mathcal{G}^*$, we proceed exactly as we did when studying the extended zero-finding problem $\mathcal{F}$. That is, we first obtain numerically an approximate zero $(\bar{\mu}, \bar{\psi})$ of $\mathcal{G}^*$, then introduce a suitable approximate inverse A^* of $D\mathcal{G}^*(\bar{\mu}, \bar{\psi})$, and finally apply Theorem 4.1 to the quasi-Newton map

$$T : (\mu, \psi) \mapsto (\mu, \psi) - A^* \mathcal{G}^*(\mu, \psi). \quad (47)$$

Remark 5.3. *This fixed point map is defined on $\mathbb{R} \times \ell_{-(s-4)}^\infty$. In order to apply Theorem 4.1, we need to control various operator norms, and in particular for operators on $\ell_{-(s-4)}^\infty$. This can easily be done, as the operator norm associated to a weighted ℓ^∞ norm is nothing but the weighted supremum of the weighted ℓ^1 -norm of the rows. That is, for any $B^* \in \mathcal{L}(\ell_{-s}^\infty)$,*

$$\|B^*\|_{\ell_{-s}^\infty} = \sup_{i \in \mathbb{N}^d} \frac{1}{\xi_i^{(s)}} \sum_{j \in \mathbb{N}^d} |B_{i,j}^*| \xi_j^{(s)}.$$

However, if there exists $B \in \mathcal{L}(\ell_s^1)$ such that B^ is the adjoint of B , then we also have the identity $\|B^*\|_{\ell_{-s}^\infty} = \|B\|_{\ell_s^1}$, which brings us exactly back to the setting of Section 4.2, where we derived all the estimates using weighted ℓ^1 spaces.*

We can take advantage of this by considering $B \in \mathcal{L}(\mathbb{R} \times \ell_s^1, \mathbb{R} \times \ell_{s-4}^1)$ given by

$$B = \begin{pmatrix} 0 & -\bar{\psi}^* \\ \bar{\varphi} & D_u F(\lambda_\circ, u_\circ) - \bar{\mu} \end{pmatrix},$$

and then construct an approximate inverse $A \in \mathcal{L}(\mathbb{R} \times \ell_{s-4}^1, \mathbb{R} \times \ell_s^1)$ of this operator B , again in the form $A = A^{\leq N} - \Delta^{-2} \Pi^{\geq N}$. In view of the identity $D\mathcal{G}^(\bar{\mu}, \bar{\psi}) = B^*$, we use the adjoint A^* of A for defining T in (47). Thus*

$$DT(\bar{\mu}, \bar{\psi}) = I - A^* D\mathcal{G}^*(\bar{\mu}, \bar{\psi}) = (I - BA)^*,$$

and all the estimates for DT necessary for Theorem 4.1 can be obtained for $I - BA$, with weighted ℓ^1 spaces.

The technical details necessary to apply Theorem 4.1 to (47) are very similar to those of Section 4.2, and are presented in Appendix C.

Acknowledgments

M.B. is supported by the ANR project CAPPS: ANR-23-CE40-0004-01. The research of E.S. was partially supported by the Simons Foundation under Award 636383, and T.W. was partially supported by the Simons Foundation under Award 581334 and the Grant SFI-MPS-TSM-00013686.

Appendix

This appendix provides technical details regarding the derivation of the computable bounds required by our computer-assisted proofs. Appendix A relates to Section 4.2, Appendix B to Section 4.3 and Appendix C to Section 5.2. We emphasize once more that the main ideas can already be understood when studying a regular steady state, and that the remaining challenges are only of technical nature.

A Computable bounds for the Ohta–Kawasaki equation

The Z^1 bounds. In order to obtain the Z^1 estimates, we have to estimate the norm of the different components of $DT(\bar{x}) = I - AD\mathcal{F}(\bar{x})$. We start with the following observation. For any $x \in \Pi^{>3N}\mathcal{X}$, $D\mathcal{F}(\bar{x})x \in \Pi^{>N}\mathcal{X}$, i.e., $D\mathcal{F}(\bar{x})\Pi^{>3N} = \Pi^{>N}D\mathcal{F}(\bar{x})\Pi^{>3N}$, and hence

$$AD\mathcal{F}(\bar{x})\Pi^{>3N} = \Delta^{-2}\Pi^{>N}D\mathcal{F}(\bar{x})\Pi^{>3N}.$$

This quantity no longer depends on the finite part $A^{\leq N}$ of A , and the term $\Delta^{-2}\Pi^{>N}$ will allow us to easily obtain explicit estimates showing that $AD\mathcal{F}(\bar{x})\Pi^{>3N}$ becomes small when N is large. On the other hand, for any $x \in \Pi^{\leq 3N}\mathcal{X}$, $D\mathcal{F}(\bar{x})x \in \Pi^{\leq 5N}\mathcal{X}$, and $AD\mathcal{F}(\bar{x})x \in \Pi^{\leq 5N}\mathcal{X}$. In particular,

$$AD\mathcal{F}(\bar{x})\Pi^{\leq 3N} = \Pi^{\leq 5N}AD\mathcal{F}(\bar{x})\Pi^{\leq 3N},$$

which is a finite dimensional operator, hence we can obtain its norm (or more precisely, the norm of its components, as required by Theorem 4.1) with a finite computation.

This motivates the introduction of two separate sets of bounds $Z_{m,i}^{1,\text{finite}}$ and $Z_{m,i}^{1,\text{tail}}$, such that, for $1 \leq m, i \leq 3$,

$$\begin{aligned} \|\pi_m D_i T(\bar{x})\Pi^{\leq 3N}\|_{B(\mathcal{X}^i, \mathcal{X}^m)} &\leq Z_{m,i}^{1,\text{finite}} \\ \|\pi_m D_i T(\bar{x})\Pi^{>3N}\|_{B(\mathcal{X}^i, \mathcal{X}^m)} &\leq Z_{m,i}^{1,\text{tail}}. \end{aligned}$$

By standard properties of ℓ^1 norms, $Z_{m,i}^1 := \max(Z_{m,i}^{1,\text{finite}}, Z_{m,i}^{1,\text{tail}})$ then satisfies (31).

As already observed above, for every $1 \leq m, i \leq 3$,

$$\|\pi_m D_i T(\bar{x})\Pi^{\leq 3N}\|_{B(\mathcal{X}^i, \mathcal{X}^m)} = \|\pi_m \Pi^{\leq 5N} D_i T(\bar{x})\Pi^{\leq 3N}\|_{B(\mathcal{X}^i, \mathcal{X}^m)},$$

which amounts to a weighted ℓ^1 operator norm of a finite matrix, therefore we can obtain all the $Z_{m,i}^{1,\text{finite}}$ terms with finite computations.

It remains to find computable tail estimates. First note that one can take $Z_{1,i}^{1,\text{tail}} = 0$ for every $i = 1, 2, 3$ (simply because, for all $x = (\lambda, u, \varphi) \in \mathcal{X}$, $\Pi^{>N}x = (0, \Pi^{>N}u, \Pi^{>N}\varphi)$, hence one has $\pi_1 \Pi^{>N}x = 0$). Moreover, $Z_{m,1}^{1,\text{tail}} = 0$ for $m = 1, 2, 3$, as

$$D_1 \mathcal{F}(\bar{x}) = D_\lambda \mathcal{F}(\bar{x}) = \begin{pmatrix} 0 \\ -\Delta(\bar{u} - \bar{u}^3) - \sigma(\bar{u} - \mu) \\ -\Delta((1 - 3\bar{u}^2)\bar{\varphi}) - \sigma\bar{\varphi} \end{pmatrix} \in \Pi^{\leq 3N}\mathcal{X},$$

hence $D_1\mathcal{F}(\bar{x})\Pi^{>3N} = 0$. We look at the remaining terms separately. We have

$$\pi_2 D_2 \mathcal{F}(\bar{x})u = D_u F(\bar{\lambda}, \bar{u})u = -\Delta^2 u - \bar{\lambda}\Delta((1 - 3\bar{u}^2)u) - \bar{\lambda}\sigma u,$$

therefore

$$\pi_2 D_2 T(\bar{x})\Pi^{>3N}u = -\bar{\lambda}\Delta^{-1}\Pi^{>N}((1 - 3\bar{u}^2)\Pi^{>3N}u) - \bar{\lambda}\sigma\Delta^{-2}\Pi^{>3N}u,$$

and

$$\|\pi_2 D_2 T(\bar{x})\Pi^{>3N}u\|_X \leq |\bar{\lambda}| \left(\frac{\|1 - 3\bar{u}^2\|_X}{\varrho_N} + \frac{|\sigma|}{\varrho_{3N}^2} \right) \|\Pi^{>3N}u\|_X, \quad (48)$$

where ϱ_N denotes the smallest eigenvalue modulus of $\Delta\Pi^{>N}$, namely

$$\varrho_N = \min_{1 \leq j \leq d} \left(\frac{\pi(N_j + 1)}{L_j} \right)^2. \quad (49)$$

We emphasize that the estimate (48) relies on the fact that $X = \ell_s^1$ is a Banach algebra, which is why we assumed $s \geq 0$. In conclusion, we take

$$Z_{2,2}^{1,\text{tail}} = |\bar{\lambda}| \left(\frac{\|1 - 3\bar{u}^2\|_X}{\varrho_N} + \frac{|\sigma|}{\varrho_{3N}^2} \right).$$

Similarly, from

$$\pi_3 D_2 \mathcal{F}(\bar{x})u = D_{uu} F(\bar{\lambda}, \bar{u})(u, \bar{\varphi}) = 6\bar{\lambda}\Delta(\bar{u}\bar{\varphi}u),$$

one obtains

$$Z_{3,2}^{1,\text{tail}} = 6|\bar{\lambda}| \frac{\|\bar{u}\bar{\varphi}\|_X}{\varrho_N}.$$

Next,

$$\pi_2 D_3 \mathcal{F}(\bar{x}) = D_\varphi F(\bar{\lambda}, \bar{u}) = 0,$$

hence we take $Z_{2,3}^{1,\text{tail}} = 0$. Finally

$$\pi_3 D_3 \mathcal{F}(\bar{x})\varphi = D_u F(\bar{\lambda}, \bar{u})\varphi,$$

and therefore we take $Z_{3,3}^{1,\text{tail}} = Z_{2,2}^{1,\text{tail}}$. This completes our derivation of the Z^1 bounds.

The Z^2 bounds. The derivation of the Z^2 estimates is more elaborate, because it involves many terms. Apart from that, however, it is very straightforward. First, observe that one has the identity $DT(x) - DT(\bar{x}) = -A(D\mathcal{F}(x) - D\mathcal{F}(\bar{x}))$, and therefore

$$\|\pi_m(D_i T(x) - D_i T(\bar{x}))\|_{B(\mathcal{X}^i, \mathcal{X}^m)} \leq \sum_{l=1}^3 \|\pi_m A \pi_l\|_{B(\mathcal{Y}^l, \mathcal{X}^m)} \|\pi_l(D_i \mathcal{F}(x) - D_i \mathcal{F}(\bar{x}))\|_{B(\mathcal{X}^i, \mathcal{Y}^l)},$$

where we go via the $\mathcal{Y}$ spaces to account for the fact that $D\mathcal{F}(x) - D\mathcal{F}(\bar{x})$ is not bounded on $\mathcal{X}$. Owing to the structure of A , each $\|\pi_m A \pi_l\|_{B(\mathcal{Y}^l, \mathcal{X}^m)}$ amounts to a finite computation, and we are left with estimating

$$\|\pi_l(D_i \mathcal{F}(x) - D_i \mathcal{F}(\bar{x}))\|_{B(\mathcal{X}^i, \mathcal{Y}^l)} \leq \sum_{j=1}^3 \sup_{x \in \text{Box}(\bar{x}, r^*)} \|\pi_l D_{ij} \mathcal{F}(x)\|_{BL(\mathcal{X}^i \times \mathcal{X}^j, \mathcal{Y}^l)} \|\pi_j(x - \bar{x})\|_{\mathcal{X}^j},$$

where $BL(\mathcal{X}^i \times \mathcal{X}^j, \mathcal{Y}^l)$ denotes the space of bounded bilinear operators from $\mathcal{X}^i \times \mathcal{X}^j$ into $\mathcal{Y}^l$. That is, we are going to find computable constants $\tilde{Z}_{l,i,j}^2$ such that, for all $1 \leq i, j, l \leq 3$,

$$\sup_{x \in \text{Box}(\bar{x}, r^*)} \|\pi_l D_{ij} \mathcal{F}(x)\|_{BL(\mathcal{X}^i \times \mathcal{X}^j, \mathcal{Y}^l)} \leq \tilde{Z}_{l,i,j}^2,$$

and then take

$$Z_{m,i,j}^2 := \sum_{l=1}^3 \|\pi_m A \pi_l\|_{B(\mathcal{Y}^l, \mathcal{X}^m)} \tilde{Z}_{l,i,j}^2.$$

There are quite a few $\tilde{Z}_{l,i,j}^2$ terms to estimate, but each of them is rather trivial. For instance, recalling that $\pi_2 \mathcal{F}(x) = \bar{F}(\lambda, u)$, we have

$$\pi_2 D_{12} \mathcal{F}(x) = D_{\lambda u} F(\lambda, u) = -\Delta((1 - 3u^2) \cdot),$$

meaning that

$$D_{\lambda u} F(\lambda, u)(\tilde{\lambda}, \tilde{u}) = -\Delta((1 - 3u^2)\tilde{u}) \tilde{\lambda}.$$

In particular, using $\|\Delta\|_{B(X,Y)} \leq \max_i \left(\frac{\pi}{L_i}\right)^2$ and $x \in \text{Box}(\bar{x}, r^*)$, one obtains

$$\begin{aligned} \|D_{\lambda u} F(\lambda, u)(\tilde{\lambda}, \tilde{u})\|_Y &\leq \max_i \left(\frac{\pi}{L_i}\right)^2 \|1 - 3u^2\|_X \|\tilde{u}\|_X |\tilde{\lambda}| \\ &\leq \max_i \left(\frac{\pi}{L_i}\right)^2 (\|1 - 3\bar{u}^2\|_X + 6 \|\bar{u}\|_X r_u^* + 3(r_u^*)^2) \|\tilde{u}\|_X |\tilde{\lambda}|, \end{aligned}$$

hence

$$\sup_{x \in \text{Box}(\bar{x}, r^*)} \|\pi_2 D_{12} \mathcal{F}(x)\|_{BL(\mathcal{X}^1 \times \mathcal{X}^2, \mathcal{Y}^2)} \leq \max_i \left(\frac{\pi}{L_i}\right)^2 (\|1 - 3\bar{u}^2\|_X + 6 \|\bar{u}\|_X r_2^* + 3(r_2^*)^2),$$

and we can take

$$\tilde{Z}_{2,1,2}^2 = \max_i \left(\frac{\pi}{L_i}\right)^2 (\|1 - 3\bar{u}^2\|_X + 6 \|\bar{u}\|_X r_2^* + 3(r_2^*)^2).$$

All of the other $\tilde{Z}_{l,i,j}^2$ terms are obtained in a similar fashion.

B Computable bounds for the SKT equation

In this appendix, we mostly emphasize the differences with the Ohta–Kawasaki equation, which are mainly due to the nonlinear diffusion terms in the SKT system, and refer the reader again to [2] for details in a simpler case.

The Y bounds. Everything is very similar to the Ohta–Kawasaki case. Because the system is now quadratic instead of cubic, $\mathcal{F}(\bar{x}) \in \Pi^{\leq 2N}$, but then $A\mathcal{F}(\bar{x}) \in \Pi^{\leq 3N}$ because of the multiplication by $\bar{\omega} \in \Pi^{\leq N}$ and $\bar{\gamma} \in \Pi^{\leq N}$ contained in A . But in the end $A\mathcal{F}(\bar{x}) \in \Pi^{\leq 3N}$ is again finite, therefore each $Y_i = \|\pi_i A\mathcal{F}(\bar{x})\|_{\mathcal{X}_i}$ can be computed.

The Z^1 bounds. We again introduce separate $Z^{1,\text{finite}}$ and $Z^{1,\text{tail}}$ bounds such that, for all choices of $1 \leq m, i \leq 5$,

$$\begin{aligned} \|\pi_m D_i T(\bar{x}) \Pi^{\leq 2N}\|_{B(\mathcal{X}^i, \mathcal{X}^m)} &\leq Z_{m,i}^{1,\text{finite}} \\ \|\pi_m D_i T(\bar{x}) \Pi^{> 2N}\|_{B(\mathcal{X}^i, \mathcal{X}^m)} &\leq Z_{m,i}^{1,\text{tail}}, \end{aligned}$$

the slightly different projection sizes being due to F only being quadratic.

Remembering that A now incorporates multiplication operators by ω and γ , we get, for every $1 \leq m, i \leq 5$,

$$\|\pi_m D_i T(\bar{x}) \Pi^{\leq 2N}\|_{B(\mathcal{X}^i, \mathcal{X}^m)} = \|\pi_m \Pi^{\leq 4N} D_i T(\bar{x}) \Pi^{\leq 2N}\|_{B(\mathcal{X}^i, \mathcal{X}^m)}.$$

Therefore, we can again obtain all the $Z_{m,i}^{1,\text{finite}}$ with finite computations.

Regarding the tail estimates, as was the case for the Ohta–Kawasaki example, many of the $Z_{m,i}^{1,\text{tail}}$ are equal to zero, and we only give the nontrivial ones below. By slightly abusing notation, we apply the ℓ_s^1 -norm entry-wise to matrices, e.g.,

$$\|\bar{\omega}\|_{\ell_s^1} := \begin{pmatrix} \|\bar{\omega}_{11}\|_{\ell_s^1} & \|\bar{\omega}_{12}\|_{\ell_s^1} \\ \|\bar{\omega}_{21}\|_{\ell_s^1} & \|\bar{\omega}_{22}\|_{\ell_s^1} \end{pmatrix}.$$

We also denote once more by ϱ_N the smallest eigenvalue modulus of $\Delta \Pi^{> N}$, given by (49). Using that ℓ_s^1 is a Banach algebra (because we assumed $s \geq 0$), one then obtains

$$\begin{aligned} \begin{pmatrix} Z_{2,2}^{1,\text{tail}} & Z_{2,3}^{1,\text{tail}} \\ Z_{3,2}^{1,\text{tail}} & Z_{3,3}^{1,\text{tail}} \end{pmatrix} &= \left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \omega \alpha \right\|_{\ell_s^1} \\ &+ \frac{1}{\varrho_N} \|\omega\|_{\ell_s^1} \left\| \begin{pmatrix} r_1 - 2a_1 \bar{u}_1 - b_1 \bar{u}_2 & -b_1 \bar{u}_1 \\ -b_2 \bar{u}_2 & r_2 - b_2 \bar{u}_1 - 2a_2 \bar{u}_2 \end{pmatrix} \right\|_{\ell_s^1}, \end{aligned}$$

$$\begin{pmatrix} Z_{4,4}^{1,\text{tail}} & Z_{4,5}^{1,\text{tail}} \\ Z_{5,4}^{1,\text{tail}} & Z_{5,5}^{1,\text{tail}} \end{pmatrix} = \begin{pmatrix} Z_{2,2}^{1,\text{tail}} & Z_{2,3}^{1,\text{tail}} \\ Z_{3,2}^{1,\text{tail}} & Z_{3,3}^{1,\text{tail}} \end{pmatrix},$$

and

$$\begin{pmatrix} Z_{4,2}^{1,\text{tail}} & Z_{4,3}^{1,\text{tail}} \\ Z_{5,2}^{1,\text{tail}} & Z_{5,3}^{1,\text{tail}} \end{pmatrix} = \|\gamma\alpha + \omega\beta\|_{\ell_s^1} + \frac{1}{\varrho_N} \|\gamma\|_{\ell_s^1} \left\| \begin{pmatrix} r_1 - 2a_1\bar{u}_1 - b_1\bar{u}_2 & -b_1\bar{u}_1 \\ -b_2\bar{u}_2 & r_2 - b_2\bar{u}_1 - 2a_2\bar{u}_2 \end{pmatrix} \right\|_{\ell_s^1} \\ + \frac{1}{\varrho_N} \|\omega\|_{\ell_s^1} \left\| \begin{pmatrix} 2a_1\bar{\varphi}_1 + b_1\bar{\varphi}_2 & b_1\bar{\varphi}_1 \\ b_2\bar{\varphi}_2 & b_2\bar{\varphi}_1 + 2a_2\bar{\varphi}_2 \end{pmatrix} \right\|_{\ell_s^1}.$$

The Z^2 bounds. These bounds can be derived exactly as explained in Appendix A, but things are even simpler here because the SKT system only has quadratic nonlinearities.

C Computable bounds for proving transcriticality

In this final appendix we sketch how to obtain the bounds necessary to apply Theorem 4.1 to (47). Compared to Section 4.2 and Appendix A, one extra difficulty that one has to deal with is that the map $\mathcal{G}^*$ depends on (λ_o, u_o) , but in practice we only have access to $(\bar{\lambda}, \bar{u})$ and to the error bounds

$$|\lambda_o - \bar{\lambda}| \leq r_\lambda, \quad \|u_o - \bar{u}\|_X \leq r_u.$$

Thus, we split $\mathcal{G}^* = \bar{\mathcal{G}}^* + \varepsilon_{\mathcal{G}^*}$, where

$$\bar{\mathcal{G}}^*(\mu, \psi) := (\psi^*(\bar{\varphi}) - 1, D_u F(\bar{\lambda}, \bar{u})^* \psi - \mu \psi),$$

and

$$\varepsilon_{\mathcal{G}^*}(\mu, \psi) := (0, (D_u F(\lambda_o, u_o) - D_u F(\bar{\lambda}, \bar{u}))^* \psi).$$

The term $\bar{\mathcal{G}}^*$ is the one for which we do most of the careful estimates, whereas $\varepsilon_{\mathcal{G}^*}$ is treated as an error term, and thus bounded more crudely.

We first collect some straightforward-to-prove but useful a priori estimates.

Lemma C.1. *For all $s \in \mathbb{R}$, and $u \in \ell_s^1$,*

$$\|\Delta u\|_{\ell_{s-2}^1} \leq \max_i \left(\frac{\pi}{L_i} \right)^2 \|u\|_{\ell_s^1}, \quad (50)$$

$$\|\Delta^{-1} \Pi^{>N} u\|_{\ell_{s+2}^1} \leq \max_i \left(\frac{L_i}{\pi} \right)^2 \left(\frac{1 + m_N}{m_N} \right)^2 \|\Pi^{>N} u\|_{\ell_s^1}, \quad (51)$$

where $m_N = \min_i N_i + 1$.

Before formulating the next auxiliary result, we need to introduce some further notation. For $A \in \mathcal{L}(\mathbb{R} \times \ell_s^1, \mathbb{R} \times \ell_t^1)$, let us denote

$$\|A\|_{s \rightarrow t} := \begin{pmatrix} |\pi_1 A \pi_1| & \|\pi_1 A \pi_2\|_{(\ell_s^1)^*} \\ \|\pi_2 A \pi_1\|_{\ell_t^1} & \|\pi_2 A \pi_2\|_{B(\ell_s^1, \ell_t^1)} \end{pmatrix},$$

and similarly for $A^* \in \mathcal{L}(\mathbb{R} \times \ell_{-t}^\infty, \mathbb{R} \times \ell_{-s}^\infty)$. We now give a computable bound for $A^* \varepsilon_{\mathcal{G}^*}$.

Lemma C.2. *Assuming $s \geq 2$, we have the estimate*

$$\begin{aligned} (\|A^* \varepsilon_{\mathcal{G}^*}\|_{s-4 \rightarrow s-4})^* &\leq \begin{pmatrix} 0 & 0 \\ 0 & r_\lambda |\sigma| \end{pmatrix} \|A\|_{s-4 \rightarrow s-4} \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & \max_i \left(\frac{\pi}{L_i} \right)^2 \left(r_\lambda \|1 - 3\bar{u}^2\|_{\ell_{s-2}^1} + 3(|\bar{\lambda}| + r_\lambda) r_u (2\|\bar{u}\|_{\ell_{s-2}^1} + r_u) \right) \end{pmatrix} \|A\|_{s-4 \rightarrow s-2}. \end{aligned} \quad (52)$$

Proof. We simply write

$$\begin{aligned} D_u F(\lambda_o, u_o) - D_u F(\bar{\lambda}, \bar{u}) &= -\lambda_o \Delta((1 - 3u_o^2) \cdot) - \lambda_o \sigma + \bar{\lambda} \Delta((1 - 3\bar{u}^2) \cdot) + \bar{\lambda} \sigma \\ &= (\bar{\lambda} - \lambda_o) \sigma + (\bar{\lambda} - \lambda_o) \Delta((1 - 3\bar{u}^2) \cdot) + 3\lambda_o \Delta((u_o - \bar{u})(u_o + \bar{u}) \cdot), \end{aligned}$$

and use Lemma C.1. $\square$

After these preparations we can finally derive computable estimates Y_m , $Z_{m,i}^1$, and $Z_{m,i,j}^2$ which satisfy assumptions (30)-(32) of Theorem 4.1, for the operator T given by (47), with the space $\mathcal{X} = \mathbb{R} \times \ell_{-(s-4)}^\infty$, the map $\mathcal{G}$, and the linear operator A introduced in Section 5.2, all under the critical assumption that $s \geq 2$.

The Y bounds. We simply split

$$\begin{aligned} T(\bar{\mu}, \bar{\psi}) - (\bar{\mu}, \bar{\psi}) &= -A^* \mathcal{G}^*(\bar{\mu}, \bar{\psi}) \\ &= -A^* \bar{\mathcal{G}}^*(\bar{\mu}, \bar{\psi}) - A^* \varepsilon_{\mathcal{G}^*}(\bar{\mu}, \bar{\psi}). \end{aligned}$$

The first term can be computed exactly, as was the case for previous Y bounds, and (52) provides us with computable upper bounds for the second term.

The Z_1 bounds. We also start by splitting

$$DT(\bar{\mu}, \bar{\psi}) = I - A^* D\bar{\mathcal{G}}^*(\bar{\mu}, \bar{\psi}) - A^* D\varepsilon_{\mathcal{G}^*}(\bar{\mu}, \bar{\psi}).$$

Note that $\varepsilon_{\mathcal{G}^*}$ is linear, hence the last term is simply equal to $-A^* \varepsilon_{\mathcal{G}^*}$, and we can again use (52) to estimate it. Regarding the $I - A^* D\bar{\mathcal{G}}^*(\bar{\mu}, \bar{\psi})$ term, we proceed as in Remark 5.3, but replacing (λ_o, u_o) by $(\bar{\lambda}, \bar{u})$. That is, we consider $\bar{B} \in \mathcal{L}(\mathbb{R} \times \ell_s^1, \mathbb{R} \times \ell_{s-4}^1)$ given by

$$\bar{B} = \left(\begin{array}{c|c} 0 & -\bar{\psi}^* \\ \hline \bar{\varphi} & D_u F(\bar{\lambda}, \bar{u}) - \bar{\mu} \end{array} \right),$$

for which we have

$$I - A^* D \bar{\mathcal{G}}^*(\bar{\mu}, \bar{\psi}) = (I - \bar{B}A)^*.$$

We can then estimate $I - \bar{B}A : \mathbb{R} \times \ell_{s-4}^1 \rightarrow \mathbb{R} \times \ell_{s-4}^1$ in a very similar manner to what was done for previous Z_1 estimates, the only difference being that A is now on the right (instead of being on the left in $I - AD\mathcal{F}(\bar{x})$). Thus, it is sufficient to use the projections $\Pi^{\leq N}$ and $\Pi^{>N}$ to split between the finite part and the tail part of Z_1 . The finite part is obtained exactly as before, and there is only one non-zero tail term, namely

$$(I + (D_u F(\bar{\lambda}, \bar{u}) - \bar{\mu}) \Delta^{-2}) \Pi^{>N} = -\bar{\lambda} \Delta ((1 - 3\bar{u}^2) \Delta^{-2} \Pi^{>N}) - (\bar{\lambda} \sigma + \bar{\mu}) \Delta^{-2} \Pi^{>N}.$$

Using Lemma C.1, and the fact that ℓ_{s-2}^1 is a Banach algebra (it is exactly for this step that we assumed $s \geq 2$), one obtains

$$\begin{aligned} \|\Delta ((1 - 3\bar{u}^2) \Delta^{-2} \Pi^{>N})\|_{B(\ell_{s-4}^1, \ell_{s-4}^1)} &\leq \max_i \left(\frac{\pi}{L_i} \right)^2 \|1 - 3\bar{u}^2\|_{\ell_{s-2}^1} \|\Delta^{-2} \Pi^{>N}\|_{B(\ell_{s-4}^1, \ell_{s-2}^1)} \\ &\leq \left(\frac{\max_i L_i}{\min_i L_i} \right)^2 \left(\frac{1 + m_N}{m_N} \right)^2 \|1 - 3\bar{u}^2\|_{\ell_{s-2}^1} \|\Delta^{-1} \Pi^{>N}\|_{B(\ell_{s-4}^1, \ell_{s-4}^1)} \\ &\leq \left(\frac{\max_i L_i}{\min_i L_i} \right)^2 \left(\frac{1 + m_N}{m_N} \right)^2 \frac{\|1 - 3\bar{u}^2\|_{\ell_{s-2}^1}}{\varrho_N}, \end{aligned}$$

with ϱ_N introduced in Section 4.2.3. Thus, we get

$$\|(I + (D_u F(\bar{\lambda}, \bar{u}) - \bar{\mu}) \Delta^{-2}) \Pi^{>N}\|_{B(\ell_{s-4}^1, \ell_{s-4}^1)} \leq Z_{2,2}^{1,\text{tail}},$$

with

$$Z_{2,2}^{1,\text{tail}} = |\bar{\lambda}| \left(\frac{\max_i L_i}{\min_i L_i} \right)^2 \left(\frac{1 + m_N}{m_N} \right)^2 \frac{\|1 - 3\bar{u}^2\|_{\ell_{s-2}^1}}{\varrho_N} + \frac{|\bar{\lambda} \sigma + \bar{\mu}|}{\varrho_N^2}.$$

The Z_2 bounds. This is trivial, as the only nonlinear term in $\mathcal{G}^*$ is $-\mu\psi$, which leads to

$$\|\pi_2 D_{12} \mathcal{G}^*(\mu, \psi)\|_{BL(\mathcal{X}^1 \times \mathcal{X}^2, \mathcal{Y}^2)} = \|\pi_2 D_{21} \mathcal{G}^*(\mu, \psi)\|_{BL(\mathcal{X}^2 \times \mathcal{X}^1, \mathcal{Y}^2)} = 1,$$

and to all the other terms being zero.

References

- [1] G. Arioli and H. Koch. Computer-assisted methods for the study of stationary solutions in dissipative systems, applied to the Kuramoto-Sivashinski equation. *Arch. Ration. Mech. Anal.*, 197(3):1033–1051, 2010.
- [2] M. Breden. Computer-assisted proofs for some nonlinear diffusion problems. *Communications in Nonlinear Science and Numerical Simulation*, 109:Paper No. 106292, 22, 2022.
- [3] M. Breden. *Computer-assisted proofs for differential equations and dynamical systems*. Habilitation thesis, Institut Polytechnique de Paris, 2025.

- [4] M. Breden, C. Kuehn, and C. Soresina. On the influence of cross-diffusion in pattern formation. *Journal of Computational Dynamics*, 8(2):213–240, 2021.
- [5] M. Breden, E. Sander, and T. Wanner. Matlab code associated to the paper “Computer-assisted proofs for some nonlinear diffusion problems”. <https://github.com/MaximeBreden/SymmetryBreakingBifurcations>, 2026.
- [6] S. Cai and Y. Watanabe. A computer-assisted method for the diblock copolymer model. *Zeitschrift für Angewandte Mathematik und Mechanik*, 99(7):e201800125, 14, 2019.
- [7] P. Chossat and R. Lauterbach. *Methods in Equivariant Bifurcations and Dynamical Systems*, volume 15 of *Advanced Series in Nonlinear Dynamics*. World Scientific, 2000.
- [8] S.-N. Chow and J. K. Hale. *Methods of Bifurcation Theory*. Springer-Verlag, New York – Berlin – Heidelberg, 1982.
- [9] M. G. Crandall and P. H. Rabinowitz. Bifurcation from simple eigenvalues. *Journal of Functional Analysis*, 8:321–340, 1971.
- [10] M. Golubitsky, I. Stewart, and D. G. Schaeffer. *Singularities and Groups in Bifurcation Theory, Volume II*. Springer-Verlag, New York – Berlin – Heidelberg, 1988.
- [11] J. Gómez-Serrano. Computer-assisted proofs in PDE: a survey. *SeMA Journal*, 76(3):459–484, 2019.
- [12] O. Hénot. RadiiPolynomial.jl. <https://github.com/OlivierHnt/RadiiPolynomial.jl>, 2021.
- [13] M. Iida, M. Mimura, and H. Ninomiya. Diffusion, cross-diffusion and competitive interaction. *Journal of mathematical biology*, 53(4):617–641, 2006.
- [14] M. Iida, H. Ninomiya, and H. Yamamoto. A review on reaction–diffusion approximation. *Journal of Elliptic and Parabolic Equations*, 4(2):565–600, 2018.
- [15] I. Johnson, E. Sander, and T. Wanner. Branch interactions and long-term dynamics for the diblock copolymer model in one dimension. *Discrete and Continuous Dynamical Systems. Series A*, 33(8):3671–3705, 2013.
- [16] S. Kamimoto, H. K. Kim, E. Sander, and T. Wanner. A computer-assisted study of red coral population dynamics. *Pure and Applied Functional Analysis*, 7(4):1307–1337, 2022.
- [17] H. Kielhöfer. *Bifurcation Theory*, volume 156 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2012.
- [18] J. Kubica, P. Zgliczyński, and P. Kalita. Pitchfork bifurcation and heteroclinic connections in the Kuramoto–Sivashinsky PDE. *Journal of Computational Dynamics*, 12(3):325–392, 2025.
- [19] C. Kuehn and C. Soresina. Numerical continuation for a fast-reaction system and its cross-diffusion limit. *SN Partial Differential Equations and Applications*, 1(2):7, 2020.
- [20] O. E. Lanford III. A computer-assisted proof of the Feigenbaum conjectures. *Bulletin of the American Mathematical Society*, 6(3):427–434, 1982.
- [21] J.-P. Lessard and A. Pugliese. Cusp bifurcations: numerical detection via two-parameter continuation and computer-assisted proofs of existence. *Discrete and Continuous Dynamical Systems - B*, 30(6):2135–2158, 2025.
- [22] J.-P. Lessard, E. Sander, and T. Wanner. Rigorous continuation of bifurcation points in the diblock copolymer equation. *Journal of Computational Dynamics*, 4(1–2):71–118, 2017.
- [23] Y. Lou and W.-M. Ni. Diffusion, self-diffusion and cross-diffusion. *Journal of Differential Equations*, 131(1):79–131, 1996.
- [24] R. E. Moore, R. B. Kearfott, and M. J. Cloud. *Introduction to interval analysis*. SIAM, 2009.
- [25] M. T. Nakao, M. Plum, and Y. Watanabe. *Numerical Verification Methods and Computer-Assisted Proofs for Partial Differential Equations*, volume 53 of *Springer Series in Computational Mathematics*. Springer Singapore, 2019.
- [26] T. Ohta and K. Kawasaki. Equilibrium morphology of block copolymer melts. *Macromolecules*, 19(10):2621–2632, 1986.

- [27] M. Plum. Existence and enclosure results for continua of solutions of parameter-dependent nonlinear boundary value problems. *Journal of Computational and Applied Mathematics*, 60(1-2):187–200, 1995.
- [28] M. Plum. Enclosures for two-point boundary value problems near bifurcation points. In *Scientific computing and validated numerics (Wuppertal, 1995)*, volume 90 of *Mathematical Research*, pages 265–279. Akademie Verlag, Berlin, 1996.
- [29] P. Rizzi, E. Sander, and T. Wanner. Inverse norm bounds for fourth-order elliptic operators and equilibrium validation for triblock copolymers. *Communications in Nonlinear Science and Numerical Simulation*, 115:Paper No. 106789, 27 pp, 2022.
- [30] P. Rizzi, E. Sander, and T. Wanner. Cyclic symmetry induced pitchfork bifurcations in the diblock copolymer model. *Discrete and Continuous Dynamical Systems, Series B*, 29(2):666–701, 2024.
- [31] S. Rump. INTLAB - INTerval LABoratory. In T. Csendes, editor, *Developments in Reliable Computing*, pages 77–104. Kluwer Academic Publishers, Dordrecht, 1999. <http://www.tuhh.de/ti3/rump/>.
- [32] E. Sander and T. Wanner. Validated saddle-node bifurcations and applications to lattice dynamical systems. *SIAM Journal on Applied Dynamical Systems*, 15(3):1690–1733, 2016.
- [33] E. Sander and T. Wanner. Equilibrium validation in models for pattern formation based on Sobolev embeddings. *Discrete and Continuous Dynamical Systems, Series B*, 26(1):603–632, 2021.
- [34] E. Sander and T. Wanner. *Theory and Numerics of Partial Differential Equations*. SIAM, Philadelphia, in preparation, 1032 pages.
- [35] N. Shigesada, K. Kawasaki, and E. Teramoto. Spatial segregation of interacting species. *Journal of theoretical biology*, 79(1):83–99, 1979.
- [36] W. Tucker. *Validated numerics: a short introduction to rigorous computations*. Princeton University Press, 2011.
- [37] J. B. van den Berg, M. Breden, and R. Sheombarsing. Validated integration of semilinear parabolic PDEs. *Numerische Mathematik*, 156(4):1219–1287, 2024.
- [38] J. B. van den Berg and J.-P. Lessard. Rigorous numerics in dynamics. *Notices Amer. Math. Soc.*, 62(9), 2015.
- [39] J. B. Van den Berg, J.-P. Lessard, and E. Queirolo. Rigorous verification of Hopf bifurcations via desingularization and continuation. *SIAM Journal on Applied Dynamical Systems*, 20(2):573–607, 2021.
- [40] J. B. van den Berg and J. F. Williams. Validation of the bifurcation diagram in the 2D Ohta-Kawasaki problem. *Nonlinearity*, 30(4):1584–1638, 2017.
- [41] J. B. van den Berg and J. F. Williams. Rigorously computing symmetric stationary states of the Ohta-Kawasaki problem in three dimensions. *SIAM Journal on Mathematical Analysis*, 51(1):131–158, 2019.
- [42] J. B. van den Berg and J. F. Williams. Optimal periodic structures with general space group symmetries in the Ohta-Kawasaki problem. *Physica D*, 415:Paper No. 132732, 23 pp, 2021.
- [43] T. Wanner. Topological analysis of the diblock copolymer equation. In Y. Nishiura and M. Kotani, editors, *Mathematical Challenges in a New Phase of Materials Science*, volume 166 of *Springer Proceedings in Mathematics & Statistics*, pages 27–51. Springer-Verlag, 2016.
- [44] T. Wanner. Computer-assisted equilibrium validation for the diblock copolymer model. *Discrete and Continuous Dynamical Systems, Series A*, 37(2):1075–1107, 2017.
- [45] T. Wanner. Computer-assisted bifurcation diagram validation and applications in materials science. *Proceedings of Symposia in Applied Mathematics*, 74:123–174, 2018.
- [46] T. Wanner. Validated bounds on embedding constants for Sobolev space Banach algebras. *Mathematical Methods in the Applied Sciences*, 41(18):9361–9376, 2018.
- [47] P. Zgliczynski. Steady state bifurcations for the Kuramoto–Sivashinsky equation: a computer assisted proof. *J. Comput. Dyn*, 2(1):95–142, 2015.