Chapter 7
Quasiperiodicity: Rotation Numbers

Suddhasattwa Das, Yoshitaka Saiki, Evelyn Sander, and James A. Yorke

Abstract A map on a torus is called "quasiperiodic" if there is a change of variables which converts it into a pure rotation in each coordinate of the torus. We develop a numerical method for finding this change of variables, a method that can be used effectively to determine how smooth (i.e., differentiable) the change of variables is, even in cases with large nonlinearities. Our method relies on fast and accurate estimates of limits of ergodic averages. Instead of uniform averages that assign equal weights to points along the trajectory of N points, we consider averages with a non-uniform distribution of weights, weighing the early and late points of the trajectory much less than those near the midpoint $N/2$. We provide a one-dimensional quasiperiodic map as an example and show that our weighted averages converge far faster than the usual rate of $O(1/N)$, provided f is sufficiently differentiable. We use this method to efficiently numerically compute rotation numbers, invariant densities, conjugacies of quasiperiodic systems, and to provide evidence that the changes of variables are (real) analytic.

7.1 Introduction

Let X a topological space with a probability measure μ and $T : X \to X$ be a measure preserving map. Let $f : X \to E$ be an integrable function, where E is a finite-dimensional real vector space. Given a point x in X, we will refer to the

S. Das (✉)
Department of Mathematics, University of Maryland, College Park, MD, USA
e-mail: sdas11@umd.edu

Y. Saiki
Graduate School of Commerce and Management, Hitotsubashi University, Tokyo, Japan

E. Sander
Department of Mathematical Sciences, George Mason University, Fairfax, VA, USA

J.A. Yorke
Institute for Physical Science and Technology, University of Maryland, College Park, MD, USA

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long-time average of the function f along the trajectory at x

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n(x)), \tag{7.1}$$

as a **Birkhoff average**. The Birkhoff Ergodic Theorem (see Theorem 4.5.5. in [1]) states that if $f \in L^1(X,\mu)$, then (7.1) converges to the integral $\int_X f d\mu$ for μ-a.e. point $x \in X$. The Birkhoff average (7.1) can be interpreted as an approximation to an integral, but convergence is very slow, as given below.

$$\left|\frac{1}{N}\sum\nolimits_{n=1}^{N} f(T^n(x)) - \int_X f d\mu\right| \le CN^{-1},$$

and even this slow rate will occur only under special circumstances such as when $(T^n(x))$ is a quasiperiodic trajectory. In general, the rate of convergence of these sums can be arbitrarily slow, as shown in [2].

The speed of convergence is often important for numerical computations. Instead of weighing the terms $f(T^n(x))$ in the average equally, we weigh the early and late terms of the set $1,\cdots,N$ much less than the terms with $n \sim N/2$ in the middle. We insert a weighting function w into the Birkhoff average, which in our case is the following C^∞ function that we will call the **exponential weighting**

$$w(t) = \begin{cases} \exp\left(\frac{1}{t(t-1)}\right) & \text{for } t \in (0,1) \\ 0 & \text{for } t \notin (0,1). \end{cases} \tag{7.2}$$

Let $\mathbb{T}^{\boldsymbol{d}}$ denote a d-dimensional torus. For $X = \mathbb{T}^d$ and a continuous f and for $\phi \in \mathbb{T}^d$, we define what we call a **Weighted Birkhoff (WB$_N$) average**

$$\mathrm{WB}_N(f)(x) := \frac{1}{A_N}\sum_{n=0}^{N-1} w\left(\frac{n}{N}\right) f(T^n x), \text{ where } A_N := \sum_{n=0}^{N-1} w\left(\frac{n}{N}\right). \tag{7.3}$$

Note that the sum of the terms $w(n/N)/A_N$ is 1, that w and all of its derivatives are 0 at both 0 and 1, and that $\int_0^1 w(x)dx > 0$.

Quasiperiodicity Each $\vec{\rho} \in (0,1)^d$ defines a **rotation**, i.e. a map $T_{\vec{\rho}}$ on the d-dimensional torus $\mathbb{T}^d$, defined as

$$T_{\vec{\rho}} : \theta \mapsto \theta + \vec{\rho} \mod 1 \text{ in each coordinate.} \tag{7.4}$$

This map acts on each coordinate θ_j by rotating it by some angle ρ_j. We call the ρ_j values "**rotation numbers**."

A vector $\vec{\rho} = (\rho_1, \ldots, \rho_d) \in \mathbb{R}^d$ is said to be **irrational** if there are no integers k_j for which $k_1\rho_1 + \cdots + k_n\rho_n \in \mathbb{Z}$, except when all k_j are zero. In particular, this implies that each ρ_j must be irrational. The rotation numbers depend on the choice of the coordinate system. In any other coordinates in which the system is also a rotation, the rotation vector $\vec{\rho}$ is $A\vec{\rho}$, for some matrix A whose entries are integers such that the determinant of A is ± 1. Conversely, any such matrix corresponds to a coordinate change which also changes $\vec{\rho}$ to $A\vec{\rho}$.

A map $T : X \to X$ is said to be **d-dimensionally C^m quasiperiodic** on a set $X_0 \subseteq X$ for some $d \in \mathbb{N}$ iff there is a C^m-diffeomorphism $h : \mathbb{T}^d \to X_0$, such that,

$$T(h(\theta)) = h(T_{\vec{\rho}}(\theta)). \tag{7.5}$$

where $T_{\vec{\rho}}$ is an irrational rotation. In this case, h is a conjugacy of T to $T_{\vec{\rho}}$. In particular, a (pure) **irrational rotation**, (a rotation by an irrational vector $\vec{\rho}$) is a quasiperiodic map.

Invariant Measure for Quasiperiodic Maps An irrational rotation $T_{\vec{\rho}} : \mathbb{T}^d \to \mathbb{T}^d$ on the torus has a unique invariant measure, which is the Lebesgue probability measure. This measure also turns out to be the unique ergodic measure. It follows that if a dynamical system $T : X_0 \to X_0$ is d-dimensionally C^1 quasiperiodic, there is a unique T-invariant measure on X_0 which, under change of variables, becomes the Lebesgue probability measure on $\mathbb{T}^d$.

Diophantine Rotations An irrational vector $\vec{\rho} \in \mathbb{R}^d$ is said to be **Diophantine** if for some $\beta > 0$ it is **Diophantine of class** β (see [3], Definition 3.1), which means there exists $C_\rho > 0$ such that for every $\vec{k} \in \mathbb{Z}^d$, $\vec{k} \neq 0$ and every $p \in \mathbb{Z}$,

$$|\vec{k} \cdot \vec{\rho} - p| \geq \frac{C_\rho}{\|k\|^{d+\beta}}. \tag{7.6}$$

For every $\beta > 0$ the set of Diophantine vectors of class β have full Lebesgue measure in $\mathbb{R}^d$ (see [3], 4.1). The Diophantine class is crucial in the study of quasiperiodic behavior, for example in [4, 5].

Continued Fractions Every irrational number $\alpha_0 \in (0,1)$ has a representation known as its continued fraction expansion $[n_1, n_2, n_3, \ldots]$, where $n_1, n_2, n_3, \ldots$ are positive integers. It can be defined inductively as follows

$$n_1 = \lfloor \frac{1}{\alpha_0} \rfloor; \alpha_1 := \frac{1}{\alpha_0} - n_1;$$
$$n_{k+1} := \lfloor \frac{1}{\alpha_k} \rfloor; \alpha_{k+1} := \frac{1}{\alpha_k} - n_{k+1}.$$

Continued Fractions as Approximations The **k-th convergent** of an irrational $\alpha_0 \in (0,1)$ is the number p_k/q_k defined as follows.

$$\frac{p_k}{q_k} = [n_1, \ldots, n_k] := \frac{1}{n_1 + \frac{1}{\ldots + \frac{1}{a_k}}}. \tag{7.7}$$

Then for every integers $q, k \geq 0$, integer p, if $q\alpha - p$ is strictly between $q_k\alpha - p_k$ and $q_{k+1}\alpha - p_{k+1}$, then either $q \geq q_k + q_{k+1}$ or both p, q must be zero. In other words, the best approximation of α by a fraction p/q with q not exceeding q_k, is the k-th convergent p_k/q_k. We rely on the continued fraction expansion of a number to decide whether it is rational or not. Every rational number has a finite number of terms in its continued fraction expansion. If α is irrational, then the sequence continues forever, while if it is rational, it stops when some α_k is zero.

The Diophantine class β of an irrational number is a measure of how closely it can be approximated by a rational number. The Diophantine class of an irrational number can be deduced from its continued fractions. This is because the k-th convergent p_k/q_k provides the best rational approximation among all rational numbers whose denominator is $\leq q_k$.

We will now state our main theorem about fast convergence of weighted Birkhoff sums (7.3). We will first define a notion of fast convergence called super-convergence.

Definition Let $(z_N)_{N=0}^{\infty}$ be a sequence in a normed vector space such that $z_N \to z$ as $N \to \infty$. We say (z_N) has **super-polynomial convergence** to z or **super converges** to z if for each integer $m > 0$ there is a constant $C_m > 0$ such that

$$|z_N - z| \leq C_m N^{-m} \text{ for all m.}$$

Theorem 7.1.1 *Let X be a C^∞ manifold and $T : X \to X$ be a d-dimensional C^∞ quasiperiodic map on $X_0 \subseteq X$, with invariant probability measure μ. Assume T has a Diophantine rotation vector. Let $f : X \to E$ be C^∞, where E is a finite-dimensional, real vector space. Assume w is the exponential weighting (see Eq. (7.2)). Then for each $x_0 \in X_0$, the weighted Birkhoff average $WB_N f(x_0)$ has super convergence to $\int_{X_0} f d\mu$.*

Other Studies on Weighted Averages The convergence of weighted ergodic sums has been discussed, for example, [6–8]), but without any conclusions on the rate of convergence. In [9], a convergence rate of $O(N^{-\alpha})$, $(0 < \alpha < 1)$, was obtained for functionals in $L^{2+\epsilon}$ for a certain choice of weights. A series of our applications of the method discussed in this paper appear in [10], and the details of the proof of our theorem appears in [11].

The use of a temporal weight in ergodic averages has been a subject of study for several decades, usually using more generic weighting sequences in the form of

$$T_N(f) := \sum_{n=0}^{\infty} \nu_N(n) U^n(f), \text{ where } \nu_N \text{ is a probability distribution on } \mathbb{N}. \tag{7.8}$$

In our theorem, the probability measure ν_N are the values of the weight function w sampled at the points $\{n/N : 0 \leq n < N\}$ and divided by the normalizing constant A_N, as defined in (7.3). In [6], sufficient conditions were derived for (7.8) to converge in weighting sequences of a similar kind. Equations (7.3) and (7.8) arise from the study of functionals on the Hilbert Space L^2. On the other hand, Berkson and Gillespie [12] considered the convergence of (7.8) for invertible operators on Banach spaces. It was shown that for a particular choice for $(\nu_N)_{N\in\mathbb{N}}$, the operators converge in the strong operator topology to an idempotent operator.

Remark Our results apply to C^m or smooth functions, which are L^2, and carry the assumption that the underlying dynamics is quasiperiodic. We are interested in exploring the applicability of the theorem to other dynamical systems, while keeping in mind that various counter-examples exist in which weighted ergodic averages do not converge. For example, in [13], the authors derived a property called *strong sweeping property* for the operators in (7.8), under the assumption that each ν_N is a *dissipative probability measure* and certain other conditions on the underlying dynamical system (X, T). The strong sweeping out property implies that the limits do not converge but attain values over an interval of numbers. In [14] similar results are obtained to prove the lack of convergence of (7.8) for a dense set of L^1 characteristic functions, in the context of ergodic rotations of the unit circle.

7.2 Application I of Theorem 7.1.1: Rotation Numbers

To illustrate some applications of Theorem 7.1.1, we will work with the following dynamical system for the rest of the paper.

A Cylinder-Map Consider the infinitely long cylinder $\mathbb{R} \times S^1$, where S^1 is the standard topological circle. Consider the following map on this cylinder, first studied in [15].

$$\begin{aligned} x_{n+1} &= 3x_n + \sigma(x_n, y_n) \\ y_{n+1} &= y_n - \delta\sin(y_n) + \epsilon(1-\cos(x_n)) \mod 2\pi. \end{aligned} \tag{7.9}$$

Here σ is a small perturbation term, δ and ϵ are parameters satisfying $0 < 2\delta < \epsilon$. It turns out that for every such parameter value, if σ is sufficiently small, then there exists an invariant topological circle. Note that if $\sigma \equiv 0$, then this is the circle whose points are $\{(\pi, y) : y \in S^1\}$. Though the map is C^∞, the invariant circle may not be smooth. We are however interested in demonstrating that the dynamics on it is C^∞-conjugate to a rotation. See Fig. 7.1 for some of these curves.

7.2.1 Rotation Number as a Weighted Birkhoff Sum

Rotation Number Let $\bar{F} : \mathbb{R}^d \to \mathbb{R}^d$ be the lift of a quasiperiodic map $F : \mathbb{T}^d \to \mathbb{T}^d$. It is well known (see for example, [16]) that the following limit exists and is a

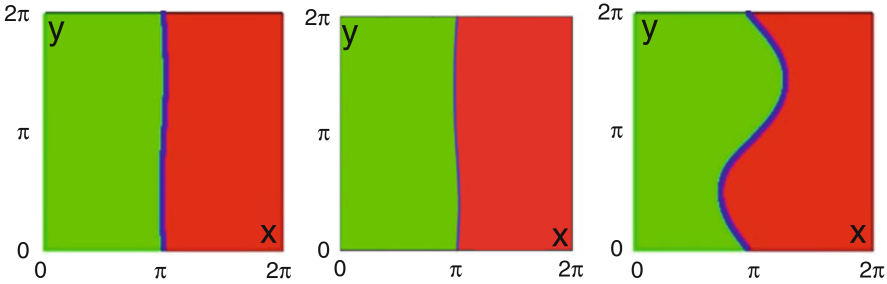

Fig. 7.1 Invariant circles in the cylinder map (7.9), for values of $(\sigma, \delta, \epsilon)$ equal to (**a**) (0.1, 0.1, 0.1), (**b**) (0.2, 0.8, 0.8) and (**c**) (1.0, 0.1, 0.1). Points in the region on the right of the *curves* diverge to $x = +\infty$, while points on the left diverge to $x = -\infty$. Therefore, these *circles* are quasiperiodic repellers and we are interested in the classification of the dynamics on these curves as periodic or quasiperiodic

constant independent of $\vec{z} \in \mathbb{R}^d$.

$$\vec{\rho}(F) := \lim_{n\to\infty} \frac{\bar{F}^n(z) - \vec{z}}{n}. \tag{7.10}$$

This limit is called the **rotation number** of F. The limit in (7.10) is a means of approximating ρ, but its convergence is bounded by the $O(1/N)$, where N is the number of iterates taken into account. We propose a better method based on the weighting factor w.

Note that in the example under discussion, X_0 is a one-dimensional quasiperiodic curve embedded in $X = \mathbb{R}^2$. Let X_0 be given the coordinates θ of a circle S^1 (in this case, θ could be the Y-coordinate of each point on the invariant curve divided by 2π). Given two angles $\theta_1, \theta_2 \in [0, 1)$, $\theta_2 - \theta_1$ denotes the positive angle difference between these two angles, i.e., with value in $[0, 1)$. We are interested in the limit $\rho := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} [\theta_{n+1} - \theta_n]$, which can be obtained as the super-convergent limit of

$$\mathrm{WB}_N((\theta_{n+1} - \theta_n)) := \frac{1}{A_N} \sum_{n=0}^{N-1} w\left(\frac{n}{N}\right) [\theta_{n+1} - \theta_n].$$

More generally, let X_0 be a quasiperiodic curve embedded in $X = \mathbb{R}^2$. Let $C := C_B \cup C_U$ be the complement of X_0 in $\mathbb{R}^2$, where C_B and C_U are the bounded and unbounded components of C respectively. For $p \in \mathbb{R}^2$, define $\phi(\theta) = (\theta - p)/\|\theta - p\|$. Therefore $\phi(\theta) \in S^1$. Let $\bar{\phi} : \mathbb{R} \to \mathbb{R}$ be the lift of ϕ. If $p \in C_B$, then $\bar{\phi}$ is of the form

$$\bar{\phi}(\bar{\theta}) = \pm\bar{\theta} + \bar{g}(\bar{\theta}),$$

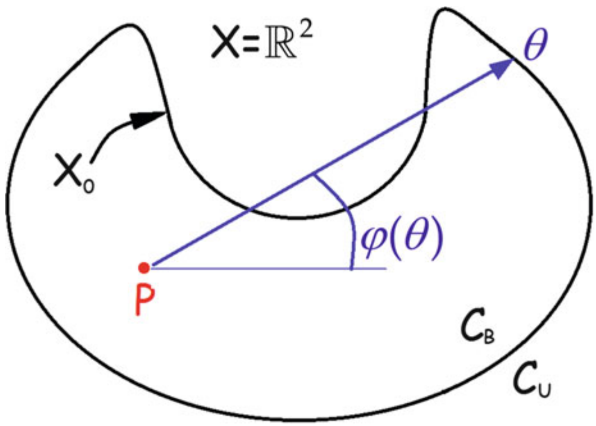

Fig. 7.2 Rotation number on a quasiperiodic curve. The numbers $\phi_n = \phi(\theta_n)$ can be used to calculate the rotation number, as stated in Application 1

where $\bar{\theta} \in \mathbb{R}$ is a lift of $\theta \in C$. Notice that the real valued function $\bar{g} : \mathbb{R} \to \mathbb{R}$ is period one and hence factors into a smooth function $g : X_0 \to \mathbb{R}$. Define a limit ρ_ϕ as follows.

$$\rho_\phi := \text{WB}_N(g(\theta)) = \frac{1}{A_N} \sum_{n=0}^{N-1} w\left(\frac{n}{N}\right) [(\theta_{n+1} - \theta_n) + g(\theta_n)].$$

Then ρ_ϕ is ρ or $1 - \rho$, depending on the orientation of θ, both being legitimate representations of ρ. We have illustrated this construction in Fig. 7.2. If $p \in C_U$, then $\rho_\phi = 0$.

7.2.2 *Error Bound for the Unweighted Method*

Given a one-dimensional quasiperiodic trajectory (x_n) on the circle $S^1 = [0, 1)$, one can define a trajectory on the real line $\bar{x}_n$ for $n = 0, \cdots, N$, where $\bar{x}_0 = x_0$, $\bar{x}_n$ is a lift of x_n and $\bar{x}_{n+1} - \bar{x}_n \in (0, 1)$. It therefore follows that $\bar{x}_{n+1} = \bar{F}(\bar{x}_n)$. Let

$$k_n := \bar{x}_n - x_n \tag{7.11}$$

be the winding number of the n-th iterate. Let the (x_n) iterates be sorted in increasing order as

$$x_{n_0} = 0 < x_{n_1} < \ldots < x_{n_N} < 1.$$

If ρ is the true rotation number, then the iterates $\theta_n = n\rho \mod 1$, for $n = 0, \ldots, N$ have the same cyclic order as the x-orbit. In other words, $0 = \theta_{n_0} < \theta_{n_1} < \ldots < \theta_{n_N}$. We can determine the interval of ρ values for which that is true. First note that

$$0 < x_{n_1} \text{ so } \rho < k_{n_1}/n_1$$
$$x_{n_N} < 1 \text{ so } \rho > (k_{n_N} + 1)/n_N.$$

Suppose $n_i < n_{i+1}$, then $(n_{i+1} - n_i)\rho = k_{n_{i+1}} - k_{n_i} + \epsilon_{n_i}$, for some $\epsilon_{n_i} \in [0, 1)$. Similarly, if $n_i > n_{i+1}$, then $(n_i - n_{i+1})\rho = k_{n_i} - k_{n_{i+1}} - \epsilon_{n_i}$. These two identities give the following two inequalities respectively.

$$\rho > \frac{k_{n_{i+1}} - k_{n_i}}{n_{i+1} - n_i}, \tag{7.12}$$

$$\rho < \frac{k_{n_i} - k_{n_{i+1}}}{n_i - n_{i+1}}. \tag{7.13}$$

For each of the $N-1$ consecutive pairs $(x_{n_i}, x_{n_{i+1}})$, we get such an inequality and they combine to give the possible range of values of ρ. Note that instead of consecutive x-s from the sorted list, we could have taken distant x-s, but the following inequality shows that would not have yielded a sharper bound.

$$\text{If } a_1, a_2, b_1, b_2 > 0, \text{ then } \frac{a_1 + a_2}{b_1 + b_2} \text{ lies in-between } \frac{a_1}{b_1} \text{ and } \frac{a_2}{b_2}. \tag{7.14}$$

7.2.3 Another Calculation of the Rotation Number Using Unweighted Birkhoff Sums

Let $F : \mathbb{T}^d \to \mathbb{T}^d$ be a homeomorphism, where $\mathbb{T}^d$ is the n-torus, obtained from the n-cube $[0, 1)^d$ by taking each coordinate modulo 1. Using the weighting methods, an initial estimate $\vec{\rho}'$ of the rotation number $\vec{\rho}$ of F, by analysing a dense trajectory $\vec{z_0}, \ldots, \vec{z_{N-1}}$. This section describes how to obtain a better estimate $\vec{\rho}''$ of $\vec{\rho}$ from $\vec{\rho}'$.

Let $\vec{z}_{n_1}, \vec{z}_{n_2}, \ldots, \vec{z}_{n_{d+1}}$ be $d + 1$ points on the trajectory which are close to the origin O and whose convex hull contains O. Then there are constants $\alpha_i \in (0, 1)$, for $i = 1, \ldots, n + 1$ such that O is a convex combination of the points $\vec{z_{n_i}}$, i.e.,

$$\vec{0} = \sum_{i=1\ldots,d+1} \alpha_i \vec{z}_{n_i}. \tag{7.15}$$

Since the map is quasiperiodic, there is a homeomorphism $G : \mathbb{T}^d \to \mathbb{T}^d$ such that for every $k = 0, \ldots, d + 1$, $\vec{z_k} = G(k\vec{\rho} \mod 1)$. If the points $\vec{z}_{n_1}, \vec{z}_{n_2}, \ldots, \vec{z}_{n_{d+1}}$ are very close to the origin, G can be considered to be linear in a neighborhood containing these points. for every $i = 1, \ldots, n+1$. Therefore, $\vec{z_{n_i}} = G(n_i\vec{\rho} \mod 1) \approx dG(0)(n_i\vec{\rho} \mod 1)$. If both sides are multiplied by $dG(0)^{-1}$ then, (7.15) becomes

$$\vec{0} \approx \sum_{i=1\ldots,d+1} \alpha_i dG(0)(n_i\vec{\rho} \mod 1). \tag{7.16}$$

Now let the integral part of $n_i\vec{\rho}$ be $\vec{k_i}$, i.e., $n_i\vec{\rho} = \vec{k_i} + \epsilon_i$, where $\vec{k_i}$ is a vector with integer entries and the entries of ϵ_i lie in $(-0.5, 0.5)^d$ and are very small. Therefore $n_i\vec{\rho} \mod (2\pi) = \epsilon_i$. Therefore (7.16) becomes

$$\vec{0} = \sum_{i=1\ldots,d+1} [\alpha_i(n_i\vec{\rho} - \vec{k_i})]. \tag{7.17}$$

Therefore, the equation can be solved to ρ as

$$\vec{\rho} = \frac{\sum_{i=1,\ldots,d+1} \alpha_i \vec{k_i}}{\sum_{i=1,\ldots,d+1} \alpha_i n_i}. \tag{7.18}$$

Note that for every $i = 1, \ldots, d+1$, $\vec{k_i}/n_i$ is a close approximation to ρ, so the sum (7.18) is an optimal combination of these optimizations.

7.2.4 Fine Tuning the Rotation Number

Let (x_n) be a quasiperiodic trajectory on a circle $S^1 = [0, 1)$. If we attempt to graph the conjugacy map $h(\theta)$ from (7.5), we have only N points and they are not equally spaced. We can compute the slopes between successive points and choose $\hat{\rho}$ so as to minimize the fluctuations in the derivatives of successive slopes. Define points $\theta_n = n\hat{\rho} \mod 1$. As before, let the (x_n) iterates be sorted in increasing order as

$$x_{n_0} = 0 < x_{n_1} < \ldots < x_{n_N} < 1.$$

This ordering will be the same (cyclically) as that of $\theta_0, \ldots, \theta_{N-1}$. Therefore, if consider the graph of h, the successive points of the graph are $p_j := (\theta_j, x_{n_j})$. The slope from p_j to p_{j+1} is:

$$S_i = \frac{\Delta x}{\Delta \theta} := \frac{x_{n_{i+1}} - x_{n_i}}{n_{i+1}\hat{\rho} \mod 1 - n_i\hat{\rho} \mod 1}.$$

From each estimate $\hat{\rho}$ of ρ, a circle map $h : S^1 \to S^1$ be constructed which maps $n\hat{\rho} \mapsto y_n$. From h, one can construct the map $h : S^1 \to S^1$ defined as $g(\theta) = h(\theta) - \theta$. When the function h is lifted to $\mathbb{R}$ it becomes a function with period one.

The closer $\hat{\rho}$ is to the true rotation number ρ, the smoother h is going to be. The following is used as a measure of smoothness of the h.

$$\sigma(\hat{\rho}) := \sum_{i=0,\ldots,N} \left[\left(\frac{\Delta x}{\Delta \theta} \right)_i - \left(\frac{\Delta x}{\Delta \theta} \right)_{i-1} \right]^2, \tag{7.19}$$

where the indices -1 refers to the index N. The sequence of quantities $(\Delta x/\Delta\theta)_i$ is defined as,

$$\left(\frac{\Delta x}{\Delta \theta} \right)_i := \frac{[x_{n_i} + k_{n_i} - n_i\hat{\rho}] - [x_{n_{i-1}} + k_{n_{i-1}} - n_{i-1}\hat{\rho}]}{[n_i\hat{\rho} \mod 1] - [n_{i-1}\hat{\rho} \mod 1]}, \tag{7.20}$$

where the sequence (k_n) is as in (7.11). Equation (7.19) is a measure of the smoothness of h in terms of the sum of the squares of the difference between successive slopes of the map h. If h is smooth, the slope changes slowly and the sum is expected to be small. We can change ρ to minimize the quantity $\sigma(\rho)/\rho$.

7.3 Other Applications of Theorem 7.1.1

We will now describe a computationally efficient method of determining whether invariant tori show quasiperiodic behavior, and we will numerically estimate the analyticity of the conjugacy to a pure rotation. There is a large volume of literature about determining invariant periodic or quasiperiodic sets, these being two of the three types of typical recurrent behavior. An algorithm was introduced in [17], which uses the Newton's method to determine all periodic orbits up to a fixed period along with their basins of attraction. Variants of the Newton's method have been employed to determine quasiperiodic trajectories in various other settings. For example, Becerra et al. [18] used the monodromy variant of Newton's method to locate periodic or quasi-periodic relative satellite motion. In [17], a quantity called local Lyapunov exponent distribution was defined and used to locate basins of small period/quasiperiodic trajectories which lie in the vicinity of larger quasiperiodic trajectories. This step is followed by an application of the Newton method. They used this method to locate co-existing quasiperiodic and periodic trajectories in the standard map. In [19], the authors defined an invariance equation involving partial derivatives. The invariant tori are then computed using finite element methods of PDE-s. See [19, Chap. 2] for more references on the numerical computation of invariant tori.

The analysis is based on the use of Theorem 7.1.1 for performing fast integration of smooth, periodic functions on the torus.

Application II, Computing the Integral of a Periodic C^∞ Function A C^∞ periodic map $f : \mathbb{R}^d \to E$ can be integrated with respect to the Lebesgue measure quickly and accurately in the following manner. We first rescale coordinates so that its domain is a d-dimensional torus $\mathbb{T}^d = [0,1]^d \mod 1$. We next choose any $\vec{\rho} = (\rho_1, \cdots, \rho_d) \in (0,1)^d$ of Diophantine class $\beta \geq 0$. For example, a good choice for the case $d = 1$ is $\rho = \frac{\sqrt{5}-1}{2}$, the golden ratio, for which $\beta = 0$. Let $T = T_{\vec{\rho}}$ be the rotation by the Diophantine vector ρ on $\mathbb{T}^d$. Let w be the exponential weighting function Eq. (7.2). Then by Theorem 7.1.1, for every $\theta \in \mathbb{T}^d$, $\mathrm{WB}_N(f)(\theta)$ *has super convergence to* $\int_{\mathbb{T}^d} f d\mu$ *and convergence is uniform in* θ.

7.3.1 *Application III, Fourier Series of the Embedding*

After computing the rotation number ρ by the method explained in Application 1, we can construct the parameterization $\phi = h(\theta)$, where $h : S^1 \to \mathbb{R}$, for which $x_{n+1} = T(x_n)$ is conjugate to the pure rotation $\theta_{n+1} = \theta_n + \rho$. The map h is not known explicitly, but its values $(x_n := h(n\vec{\rho} \mod 1))_{n=0,1,2,\ldots}$ are known. Let $\bar{h} : \mathbb{R} \to \mathbb{R}$ be a lift of the map h. Consider the following function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(\theta) := \bar{h}(\theta) - \theta. \tag{7.21}$$

The continuity and the degree of differentiability of h is the same as that of g, and the latter can be non-rigorously estimated by observing the rate of decay of the Fourier series coefficients of the function g. For every $k \in \mathbb{Z}$, the k-th Fourier coefficient of g is described below.

$$a_k(h) := \int_{S^1} h(\theta) e^{-i2\pi k\theta} d\theta.$$

For every $\theta \in S^1$, h has the Fourier series representation

$$h(\theta) = \sum_{k \in \mathbb{Z}} a_k e^{i2\pi k\theta}.$$

To study the decay rate of the coefficients a_k with $|k|$, we need to accurately calculate each term a_k. By Theorem 7.1.1 , $a_k(h)$ can be approximated by a weighted Birkhoff sum that has super convergence to $a_k(h)$,

$$a_k(h) = \lim_{N\to\infty} \mathrm{WB}_N[h(\theta)e^{-i2\pi k\theta}] = \lim_{N\to\infty} \sum_{n=0}^{N-1} w\left(\frac{n}{N}\right) x_n e^{-i2\pi nk\rho}.$$

Instead of computing the complex-valued Fourier coefficients, we will compute the Fourier sine and cosine series. Given a periodic map $f : S^1 \to \mathbb{R}$, the Fourier sine and cosine representation of f is the following. For every $t \in S^1$,

$$f(t) = \frac{a_0}{2} + \sum_{n=1,2,\ldots} a_n \cos(2n\pi t) + \sum_{n=0,1,2,\ldots} b_n \sin(2n\pi t), \tag{7.22}$$

where the coefficients a_n and b_n are given by the following formulas.

$$a_n = 2 \int_{\theta \in S^1} f(\theta) \cos(2n\pi\theta) d\theta, \tag{7.23}$$

$$b_n = 2 \int_{\theta \in S^1} f(\theta) \sin(2n\pi\theta) d\theta. \tag{7.24}$$

See Fig. 7.4 for the decay of the Fourier sine and cosine coefficients with k.

Role of Length of Trajectory Using a higher number of iterates enables a more accurate computation of the higher order Fourier terms (up to 400 terms), up to the accuracy limit which is possible with the precision being used. Figure 7.3 shows that the sine and cosine series decay exponentially, as expected in an analytic conjugation.

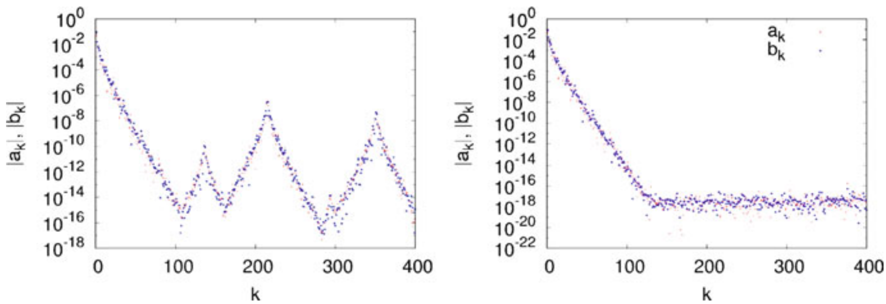

Fig. 7.3 Accuracy of Fourier series, orbit length and computer arithmetic. In all these figures, the Fourier sine and cosine terms of the map $h(\theta) - \theta$ were calculated up to 400 terms, with $\epsilon = 0.8$, $\delta = 0.8$, $\sigma = 0.2$. In (**a**) and (**b**), 10^4 and 2×10^5 iterates respectively were used along with double precision. The earlier Fig. 7.4 shows the highest accuracy, as it used 2×10^5 iterates and quadruple precision. From these results, it becomes apparent that increasing the number of iterates leads to an accurate calculation of higher order Fourier terms. Use of double precision limits the accuracy of the results to 10^{-16} while the accuracy limit for quadruple precision is around 10^{-32}, as seen is Fig. 7.4

7.3.2 Smoothness of Conjugacies

In [20], Denjoy proved that if a C^2, orientation-preserving circle diffeomorphism has an irrational rotation number α, then it is topologically conjugate to the pure rotation $T_\alpha : z \mapsto z + \alpha$, via some continuous map h. We are interested in inferring more about the smoothness class of h. The question of smoothness of conjugacy to a pure rotation is an old problem. While we have described here a non-rigorous method, the papers [3, 21–23] arrive at rigorous conclusions on the differentiability of f by making various assumptions on the smoothness of the quasiperiodic map T and the Diophantine class of its rotation number ρ. We will give a brief summary of some of the classical results before describing our approach.

The **Arnold family** is a commonly studied in the context of existence of quasiperiodic trajectories. In this seminal work [16], Arnold studied the following 2-parameter family of circle diffeomorphisms where ϕ is a T-periodic real analytic function with period one, meaning $\phi(y+1) \equiv \phi(y)$:

$$A_{\omega,\epsilon} : y \mapsto y + \omega + \epsilon\phi(y) \mod 1 \text{ for } y \in [0,1] \text{ and } \epsilon \text{ in } [0,1). \tag{7.25}$$

One of the main theorems about this generic family of maps is that was that for ω belonging to a certain, full-measure set of irrational numbers, for all small values of the parameter ϵ, the map (7.25) will be analytically conjugate to the pure rotation T_ρ (7.4). By "small" ϵ, we mean all ϵ which are less in magnitude than a positive constant ϵ_0 which depends on ω. Subsequently, several other conjugacy results have been established. They differ in their claims on the degree of smoothness of the conjugacy ($C^0, C^1, C^2, \ldots$, or C^∞ or C^ω); as well as in their assumptions on f.

Consider the following four assumptions on the circle map F which will serve as the hypothesis of some of the known results we are going to cite. The subscripted variables, namely r and ν denote parameters which are a part of their respective assumptions.

(A1) $_r$ F is C^r.

(A2) $_\nu$ $\rho(F)$ is irrational and there is some $\nu > 0$ such that the continued fraction expansion $k_1, k_2, \ldots$ of the rotation number satisfies : $\{k_n n^{-\nu} : n \in \mathbb{N}\}$ is bounded.

(A3) $_\beta$ There is $\beta \geq 0$ and a $c > 0$ such that for every $n \in \mathbb{Z} - \{0\}$, $|e^{2\pi\iota n\rho} - 1| > c|n|^{-\beta-1}$. Equivalently, ρ is Diophantine with Diophantine class β.

(A4) $\lim_{B\to\infty} \limsup_{N\to\infty} \left[\sum_{\substack{1 \leq i \leq N \\ a_i \geq B}} \ln(1+a_i) \Big/ \sum_{1\leq i\leq N} \ln(1+a_i) \right] = 0$. A4 is a full-measure condition.

In [3], Herman proves that F is C^1-conjugate to a pure rotation if it satisfies $(\text{A1})_r$ for some $r > 2$. By Katznelson and Ornstein [21], if F satisfies $(\text{A1})_r$ for some $r > 2$ and $(\text{A3})_0$, then h is absolutely continuous. According to [22] if F satisfies more generally $(\text{A1})_r$ for some $r > 2$ and $(\text{A3})_\tau$, then h is $C^{r-1-\tau-\epsilon}$ for every $\epsilon > 0$.

In [24], the following smoothness result is derived for rotation numbers belonging to a full measure subset of $\mathbb{R}$. There exists $\epsilon > 0$ and $C > 0$ such that for $\forall \beta > 0$, if F satisfies $(A1)_5$, $(A3)_\beta$ and if $\|f - R_\alpha\|_{C^5} \leq \epsilon\gamma$, then h is C^3 and satisfies

$$\|D^3 h\|_{L^2} \leq \frac{C}{\gamma}\|f - R_\alpha\|_{C^5}.$$

In [23], it is shown that if F satisfies $(A3)_\beta$ for some $\beta \geq 0$ and $(A1)_r$, for $r \geq 3$ and $r > 2\beta + 1$. Then h is $C^{r-1-\beta-\epsilon}$ for every $\epsilon > 0$. As a corollary, it follows that under the same hypothesis, if F is C^∞, then so is h.

In [25], the following conclusions are made about h:

- If F satisfies $(A1)_r$ for some $r \geq 3$ and α satisfies (A4), then h is $C^{r-1-\epsilon}$, for every $\epsilon > 0$.
- F is conjugate to a rotation if and only if the sequence $(F^n)_{n\in\mathbb{N}}$ is bounded in the C^1-topology.

In our case, we conclude that h is real analytic if $\|a_k\|$ decreases exponentially fast, i.e.,

$$\log \|a_k\| \leq A + B|k| \tag{7.26}$$

for some A and B, to the extent checkable by compute precision (see Fig. 7.4). In this section, $F : S^1 \to S^1$ is a circle diffeomorphism and $\alpha := \rho(F)$ is its rotation number.

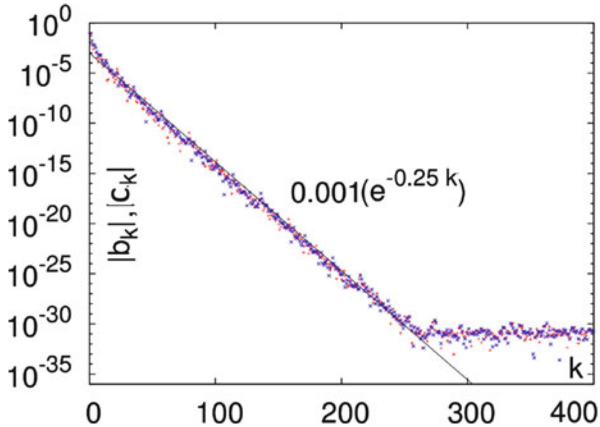

Fig. 7.4 Exponential decay of Fourier coefficients for the cylinder-map (7.9). The figure shows the magnitude of the Fourier coefficients of the periodic function g in (7.21). The first 400 Fourier sine and cosine terms were calculated and the magnitude of the n-th sine and cosine terms was plotted as a function of n, in a log(base 10)-linear scale. All calculations were carried out in quadruple precision computer arithmetic. The graph shows that the Fourier coefficients decay according to the law in (7.26), with $c = -0.25$. The tail of the graph appears flat because the higher order Fourier coefficients could not be calculated to values with magnitude less than the limits of quadruple precision

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