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COMPLEX TRANSIENT PATTERNS ON THE DISK

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Abstract. This paper studies spinodal decomposition in the Cahn-Hilliard model on the unit disk. It has previously been shown that starting at initial conditions near a homogeneous equilibrium on a rectangular domain, solutions to the linearized and the nonlinear Cahn-Hilliard equation behave indistinguishably up to large distances from the homogeneous state. In this paper we demonstrate how these results can be extended to nonrectangular domains. Particular emphasis is put on the case of the unit disk, for which interesting new phenomena can be observed. Our proof is based on vector-valued extensions of probabilistic methods used in Wanner [37]. These are the first results of this kind for domains more general than rectangular.

1. **Introduction.**

Many natural dynamical processes generate complex and intriguing patterns. In some cases, these patterns are robust in that if the underlying experiment is repeated, the same pattern occurs. The most elementary examples are stationary, i.e., they do not change with time. However, many dynamical processes produce patterns which exhibit neither robustness nor stationarity. The inherent noise present in an experiment gives rise to different patterns each time the experiment is repeated, but all patterns have the same qualitative features. One such dynamical process is spinodal decomposition: This pattern formation process occurs during the phase separation of alloys. Specifically, if a high-temperature homogeneous mixture of several metallic components is rapidly cooled below a certain temperature, a process of phase separation can set in, during which the mixture becomes inhomogeneous. It forms a fine-grained characteristic snake-like structure, with steep transition layers between the components and a characteristic length scale. If the experiment is repeated even with the greatest care to ensure almost identical initial conditions, the observed pattern is clearly distinct but with the same characteristic features.

A well-known model for spinodal decomposition in binary alloys is due to Cahn and Hilliard [6, 9]. They propose the nonlinear parabolic equation

$$u_t = -\Delta(\varepsilon^2 \Delta u + f(u)) \quad \text{in} \quad \Omega\,, \qquad \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \tag{1.1}$$

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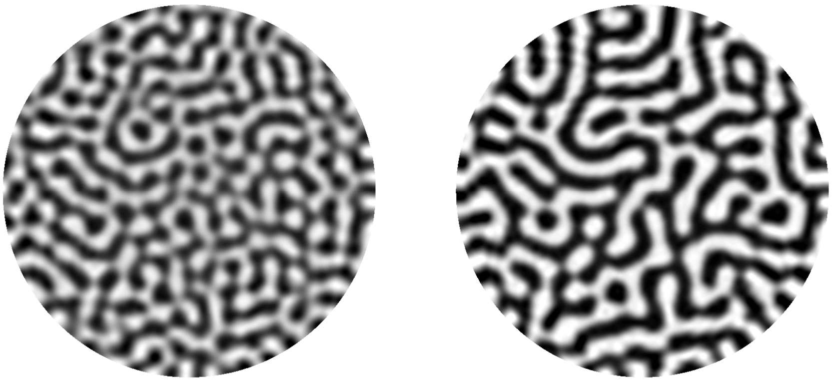

FIGURE 1. Patterns observed during spinodal decomposition on the disk for $\varepsilon = 0.015$. The left diagram shows the solution at time $t = 0.0062$, the right diagram is for $t = 0.0123$. Black represents one component ($u = -1$), and white the other ($u = 1$). Shades of gray represent a mixture of the two components.

for the concentration $u = u(t, x)$ of one of the two metals as a function of time and space, where u is affine scaled to be between -1 and 1. The domain $\Omega \subset \mathbb{R}^n$ is bounded with appropriately smooth boundary, $n \in \{1, 2, 3\}$, and the function $-f$ is the derivative of a double-well potential F, the standard example being the cubic function $f(u) = u - u^3$. The small parameter $\varepsilon > 0$ models interaction length.

The Cahn-Hilliard equation is mass-conserving; that is, the total concentration $\int_\Omega u(t, x)\, dx$ remains constant along any solution u. It is also a gradient system with respect to the standard van der Waals free energy functional (cf. Fife [18]). Details of the relationship between the model and the physical process can be found in Cahn [7, 8], Elder, Desai [14], Elder, Rogers, Desai [15], Hilliard [23], Hyde et al. [24], and Langer [27]. Numerical simulations have been done by Bai et al. [3, 4], Copetti [10], Copetti, Elliott [11], Elliott [16], Elliott, French [17], Hyde et al. [24], Nash [31], and Sander, Wanner [33].

Notice that any constant function $\bar{u}_o \equiv \mu$ is a homogeneous equilibrium for Equation (1.1). The equilibrium is unstable if μ is contained in the *spinodal interval*, which consists of the usually connected set of all $\mu \in \mathbb{R}$ for which $f'(\mu) > 0$. Thus, if μ lies in the spinodal interval, any orbit of (1.1) originating at an initial condition $u_o \approx \bar{u}_o$ is likely to be driven away from $\bar{u}_o$. Figure 1 shows the time evolution of a solution of the Cahn-Hilliard equation on the disk starting near the unstable homogeneous equilibrium with $\mu = 0$. We want to explain the occurrence of such patterns, and in particular their dependence on small ε values. Therefore, we need to understand exactly how such solutions depart from the homogeneous equilibrium. To gain such an understanding for solutions starting near the unstable equilibrium, Sander and Wanner [33] performed Monte-Carlo simulations in one space dimension for comparing the solution u of the nonlinear Cahn-Hilliard equation to the solution v of the corresponding linearization at the homogeneous equilibrium $\mu = 0$ with the same initial condition. They consider the *relative distance* $\|u - v\|/\|v\|$, where the norm is the $H^2(\Omega)$-norm. These simulations indicate that for initial conditions near $\bar{u}_o$, the solutions u and v remain very close with respect to their relative distance (bounded by an ε-independent fixed small constant C) until the maximum norm of the solution u reaches an ε-independent threshold. The latter corresponds to the $H^2(\Omega)$-norm of u being of the order ε^{-2} asymptotically as ε limits to zero. We use the notation R_ε for the $H^2(\Omega)$-norm to which the solutions u and v remain close. Simulations by Nash [31] confirm that this order of $R_\varepsilon \sim \varepsilon^{-2}$ is also true for two-dimensional rectangular domains. More recently, Desi [13] performed similar simulations for the unit disk. For these simulations he extended spectral methods for circular domains as described in [36] to the Cahn-Hilliard situation, with linearly implicit time-stepping. His results show that again the same order is recovered.

Motivated by these numerical results, consider the linearization of (1.1) at the homogeneous equilibrium $\bar{u}_o \equiv \mu$. It is given by

$$\begin{aligned} v_t &= -\Delta(\varepsilon^2 \Delta v + f'(\mu)v) =: A_\varepsilon v \quad \text{in} \quad \Omega \,, \\ \frac{\partial v}{\partial \nu} &= \frac{\partial \Delta v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \,. \end{aligned} \tag{1.2}$$

If we introduce the Hilbert space

$$X = \left\{ v \in L^2(\Omega) : \int_\Omega v \, dx = 0 \right\} \,, \tag{1.3}$$

then the operator $A_\varepsilon : X \to X$ defined in (1.2), associated with the domain

$$D(A_\varepsilon) = \left\{ v \in X \cap H^4(\Omega) : \frac{\partial v}{\partial \nu}(x) = \frac{\partial \Delta v}{\partial \nu}(x) = 0 \,,\, x \in \partial\Omega \right\} \,, \tag{1.4}$$

is self-adjoint, and $-A_\varepsilon$ is a sectorial operator; see for example [22, p. 19] or [32]. Let $0 < \kappa_1 \le \kappa_2 \le \ldots \to \infty$ denote the ordered eigenvalues of the negative Laplacian $-\Delta : X \to X$ subject to homogeneous Neumann boundary conditions, and denote the corresponding complete set of $L^2(\Omega)$-orthonormalized eigenfunctions by $\psi_1, \psi_2, \ldots$. Then the spectrum of A_ε consists of the eigenvalues

$$\lambda_{k,\varepsilon} = \kappa_k \cdot (f'(\mu) - \varepsilon^2 \kappa_k) \,, \quad k \in \mathbb{N} \,, \tag{1.5}$$

with corresponding eigenfunctions ψ_k. These eigenvalues are bounded above by

$$\lambda_\varepsilon^{\max} = \frac{f'(\mu)^2}{4\varepsilon^2} \,, \tag{1.6}$$

and asymptotically as $\varepsilon \to 0$, the largest eigenvalue grows like $\lambda_\varepsilon^{\max}$.

In the current paper, as in other previous results on the subject, we use a dynamical systems approach in the sense that we consider (1.1) as an abstract evolution equation on a suitable function space. More precisely, equation (1.1) generates a nonlinear semiflow $T_\varepsilon(t)$, $t \ge 0$, on the affine space $\mu + X^{1/2}$, where $X^{1/2}$ denotes the Hilbert space

$$X^{1/2} = \left\{ v \in H^2(\Omega) \cap X : \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\} \,. \tag{1.7}$$

The constant function $\bar{u}_o \equiv \mu$ is an equilibrium point for T_ε, and the linearization of T_ε at $\bar{u}_o$ is given by the analytic semigroup S_ε generated by A_ε.

Early results about spinodal decomposition were obtained by Grant [19] and Maier-Paape and Wanner [28, 29] by relating details of the linearization (1.2) to the evolution of solutions starting near a homogeneous equilibrium $\bar{u}_o \equiv \mu$, where μ is in the spinodal interval. The first of these early results is for one-dimensional domains. It does not explain the types of patterns seen in two and three dimensions. See the introduction of [28] for more details. The results of Maier-Paape and Wanner do apply to higher-dimensional domains. They are able to explain the ε-dependence of the characteristic thickness of patterns. As in the numerics described above, they relate solutions for the linear (1.2) and nonlinear (1.1) Cahn-Hilliard equations. Their result is that in a neighborhood V_ε of the equilibrium $\bar{u}_o$, most solutions of (1.1) starting in a smaller neighborhood $U_\varepsilon \subset V_\varepsilon$ of $\bar{u}_o$ exit V_ε close to the linear subspace

$$X_\varepsilon^+ = \operatorname{span}\{\psi_k \ : \ \lambda_{k,\varepsilon} \geq \gamma_o \cdot \lambda_\varepsilon^{\max}\} \ \subset \ X^{1/2} \,, \tag{1.8}$$

where $0 \ll \gamma_o < 1$. This *dominating subspace* is spanned by the eigenfunctions corresponding to a small percentage of the largest eigenvalues of A_ε, its dimension is proportional to $\varepsilon^{-\dim\Omega}$. Functions in X_ε^+ generally exhibit patterns similar to the one depicted in Figure 1.

The results of Maier-Paape and Wanner are not optimal in that they only describe solutions while they remain in a neighborhood of the equilibrium of size proportional to $\varepsilon^{\dim\Omega}$ with respect to the $H^2(\Omega)$-norm. In contrast, the simulations of [13, 31, 33] indicate that the characteristic patterns are observed up to a distance of order ε^{-2} in the $H^2(\Omega)$-norm. More recently, for rectangular domains, Sander and Wanner [33, 34] employed sharp nonlinearity estimates to get a better estimate on the distance to which spinodal decomposition is observed. Their result is basically as follows. For a more technical statement of the theorem, the reader is referred to the original paper.

Theorem 1.1 (Theorem 3.4 in [34]). *Consider equation (1.1) on a certain class of domains including rectangular domains, with the cubic nonlinearity $f(u) = u - u^3$ and $\mu = 0$. Let $\varrho > 0$ be arbitrarily small, but fixed, and let u_o denote an initial condition close to $\bar{u}_o \equiv \mu$, which is sufficiently close to the dominating subspace X_ε^+ defined in (1.8). Let u and v be the solutions to (1.1) and (1.2), respectively, with the same initial condition u_o. Then as long as*

$$\|u(t)\|_{H^2(\Omega)} \leq C \cdot \varepsilon^{-1+\varrho+\dim\Omega/4} \cdot \|u_o\|_{H^2(\Omega)}^{\varrho} \tag{1.9}$$

we have

$$\frac{\|u(t) - v(t)\|_{H^2(\Omega)}}{\|v(t)\|_{H^2(\Omega)}} \leq C \cdot \varepsilon^{2-\dim\Omega/2} \,. \tag{1.10}$$

That is, u remains extremely close to v until $\|u(t)\|_{H^2(\Omega)}$ exceeds the threshold given in (1.9).

While this result does not reproduce the exponent observed in the numerics of [13, 31, 33], it does provide a much tighter bound on the relative distance — leaving considerable room for improvement.

Theorem 1.1 significantly improves Maier-Paape and Wanner's results, but it is still not optimal in two ways: (*i*) It only applies to a certain class of domains. (*ii*) The above theorem is sharp in the sense that one can construct worst-case initial conditions for which the stated asymptotic estimates are exact. See [37] for more details. Yet, if we start the evolution at randomly chosen initial conditions close to $\bar{u}_o$, then the radius up to which the relative distance is of the order $O(\varepsilon^{2-\dim\Omega/2})$ is considerably larger than the one given in (1.9). Again, see [37] for more details.

Problem (*i*) arises from the fact that one of the main assumptions needed in the proof is the uniform boundedness of the $L^\infty(\Omega)$-norms of the $L^2(\Omega)$-orthonormalized eigenfunctions of the Laplacian subject to homogeneous Neumann boundary conditions. For a general domain, this is not true. The simplest domain which violates this crucial assumption is the unit disk in $\mathbb{R}^2$. These eigenfunctions and their corresponding eigenvalues are known exactly, which makes it possible to study spinodal decomposition in this setting. This is what has motivated our study of the disk.

Problem (*ii*) comes from the fact that the proof describes the behavior of all solutions within a large cone around the dominating subspace which start near the equilibrium. But typical/randomly chosen solutions do not display the worst-case behavior. This problem was solved by Wanner in [37]. He extends techniques for understanding the maximum norms of random sums from [2, 25] to understanding the growth of time varying sums of eigenfunctions of the Laplacian with Neumann boundary conditions. The following is the main result of Wanner (again, in a non-technical version):

THEOREM 1.2 (Theorem 1.2 in [37]). *Consider (1.1) with $f(u) = u - u^3$ and $\mu = 0$, and let Ω be a bounded rectangular domain in $\mathbb{R}^n$, where $n \in \{1, 2, 3\}$. Let $\varrho > 0$ be arbitrarily small, but fixed, and let u_o denote an initial condition close to $\bar{u}_o \equiv \mu$, which is sufficiently close to the dominating subspace X_ε^+ defined in (1.8). Finally, let u and v be the solutions to (1.1) and (1.2), respectively, starting at u_o. Then for most such initial conditions u_o, as long as*

$$\|u(t)\|_{H^2(\Omega)} \leq C \cdot \varepsilon^{-1+\varrho} \cdot \|u_o\|^{\varrho}_{H^2(\Omega)} , \tag{1.11}$$

we have

$$\frac{\|u(t) - v(t)\|_{H^2(\Omega)}}{\|v(t)\|_{H^2(\Omega)}} \leq C \cdot \varepsilon^2 \cdot \sqrt{|\ln \varepsilon|} . \tag{1.12}$$

This is a marked improvement on the results of Sander and Wanner, especially in dimensions two and three. Notice that in Theorem 1.2 both the bound on the relative distance and the radius to which it applies are now dimension-independent. In fact, the numerical results in [37] show that these bounds are precisely the ones realized by generic solutions, originating at randomly chosen initial conditions. At the same time, estimate (1.12) leaves considerable room for improvement. We conjecture that if the right-hand side of (1.12) is given by an ε-independent constant, then the ε-term in (1.11) is given by $\varepsilon^{-2+\varrho}$.

However, Problem (*i*) still remains. Namely, the result only applies to rectangular domains. This is due to the fact that Wanner's proof assumes the quite restrictive hypothesis that eigenfunctions are orthonormal, and in addition that all their partial derivatives are orthogonal. This is not true for most domains, in particular the disk.

In the current paper, we are able to extend the probabilistic results of Wanner [37] to the disk by developing general results for vector-valued functions, and applying these to the gradients of eigenfunctions rather than to the individual partial derivatives. In the course of doing so, we have arrived at a result which applies to general domains Ω under certain hypotheses on the growth of the $L^\infty(\Omega)$-norms of the eigenfunctions and their gradients as a function of κ_k. The following is a non-technical statement of our main result in the case of the disk.

THEOREM 1.3 (Main Result). *Consider (1.1) with $f(u) = u - u^3$ and $\mu = 0$, and let Ω be the disk. Let $\varrho > 0$ be arbitrarily small, but fixed, and let u_o denote an initial condition close to $\bar{u}_o \equiv \mu$, which is sufficiently close to the dominating subspace X_ε^+ defined in (1.8). Finally, let u and v be the solutions to (1.1) and (1.2), respectively, starting at u_o. Then for most such initial conditions u_o, as long as*

$$\|u(t)\|_{H^2(\Omega)} \leq C \cdot \varepsilon^{-3/4+\varrho} \cdot \|u_o\|^{\varrho}_{H^2(\Omega)} , \tag{1.13}$$

we have

$$\frac{\|u(t) - v(t)\|_{H^2(\Omega)}}{\|v(t)\|_{H^2(\Omega)}} \leq C \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|} . \tag{1.14}$$

Ω	Rectangle	Disk	Generic Domain (conjectured)
$\|\psi_k\|_{L^\infty(\Omega)}$	1	$\kappa_k^{1/4}$	$\ln \kappa_k$
$\||\nabla\psi_k|\|_{L^\infty(\Omega)}$	$\kappa_k^{1/2}$	$\kappa_k^{3/4}$	$\kappa_k^{1/2} \ln \kappa_k$

Table 1. Comparison of the asymptotic behavior of the maximum norms of the $L^2(\Omega)$-orthonormalized eigenfunctions ψ_k of $-\Delta$ and their gradients.

The paper proceeds as follows. In Section 2.1, we give results on the maximum norms of time-dependent superpositions of vector-valued functions with normally distributed coefficients. Particularly, we derive bounds on the probability that a superposition has a large $L^\infty(\Omega)$-norm, assuming that it is not likely to have large localized peaks; see Assumption 2.2. The verification of the latter assumption is the subject of Section 2.2, where we restrict our attention to vector-valued functions which are orthonormal with respect to the vector-valued $L^2(\Omega)$-norm. Finally, Section 2.3 sets the stage for our application to spinodal decomposition. For this, we isolate the two essential assumptions in a more specific situation. Now we consider superpositions of a finite set of eigenfunctions of the negative Laplacian subject to homogeneous Neumann boundary conditions, for arbitrary domains in dimensions 1, 2, and 3. While the eigenfunctions are automatically orthogonal, the corresponding statement for their gradients follows from Stokes' Theorem. By considering the entire vector-valued gradient at once, we are able to extend Wanner's results on random sums to establish an upper bound on the $L^\infty(\Omega)$-norm of time-varying superpositions of eigenfunctions and their gradients. These results only depend on the specific asymptotics of the $L^\infty(\Omega)$-norm of the $L^2(\Omega)$-orthonormalized eigenfunctions of the negative Laplacian and their gradients — and these do depend heavily on the underlying domain Ω, as summarized in Table 1. Note that for rectangular domains, there is a uniform bound that works for all eigenfunctions. In contrast, for the disk, the $L^\infty(\Omega)$-norm of the eigenfunctions depends on the eigenvalue κ_k. For a generic domain Ω in dimensions 1, 2, and 3 Aurich et al. [2] conjecture, based on formal computations, that the $L^\infty(\Omega)$-norm of the eigenfunctions will vary logarithmically in κ_k. A similar formal computation would yield an extra $\kappa_k^{1/2}$ factor in the $L^\infty(\Omega)$-norm for the gradient.

Remark 1.4. If one assumes that for generic domains the conjectured asymptotic behavior presented in Table 1 is correct, then our main theorem still applies. It is only modified in that (1.13) becomes

$$\|u(t)\|_{H^2(\Omega)} \le C \cdot \varepsilon^{-1+\varrho} \cdot \|u_o\|^{\varrho}_{H^2(\Omega)} ,$$

and (1.14) becomes

$$\frac{\|u(t) - v(t)\|_{H^2(\Omega)}}{\|v(t)\|_{H^2(\Omega)}} \le C \cdot \varepsilon^2 \cdot |\ln \varepsilon|^{3/2} ,$$

which basically corresponds to the case of rectangular domains.

After these general results, Section 3 concentrates on the disk to derive the precise asymptotic estimates necessary for employing the results of Section 2. We state the well-known exact formulas for these eigenfunctions in polar coordinates using products of trigonometric functions and Bessel functions, and establish the $L^\infty(\Omega)$-norm bounds on the eigenfunctions and their gradients stated above.

Section 4 applies all these results to the Cahn-Hilliard equation on a disk, and we are able to establish our main theorem. In this application, the finite set of eigenfunctions consists of the basis of the dominating subspace (1.8) for suitable γ_o. Thus, the κ_k-dependence of the $L^\infty(\Omega)$-norms translates to an ε-dependence of the radius to which linear behavior can be observed. Our main theorem gives nearly linear behavior up to distances proportional to $\varepsilon^{-3/4}$, whereas numerics indicates an order of $\varepsilon^{-1.3}$. Based on numerical simulations that will be presented at the end of Section 4, we believe that the level of symmetry of the disk leads to non-generic cancellations, the analysis of which would involve detailed calculations using the specific form of the eigenfunctions via trigonometric functions and Bessel functions.

2. **Maximum Norms of Random Sums.**

One of the main disadvantages of the results in [37] is the fact that they could only be applied to rectangular domains. This restriction is a consequence of the L^∞-gradient estimates which are necessary for studying transient patterns in the Cahn-Hilliard model. In this section, we will remove the above domain restriction. For this, we have to extend the abstract results on maximum norms of random sums in [37, Section 3] from the scalar-valued to the vector-valued case. This will be accomplished in Sections 2.1 and 2.2 below, and will allow us to directly treat the gradient estimates. The latter goal will be accomplished in Section 2.3. The results of this section apply to arbitrary bounded domains $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary. In fact, it is enough to assume that Ω is a Lipschitz domain.

2.1. **Random Time-Dependent Sums.**

All of the random sums which are considered in the following are linear combinations of certain vector-valued basis functions, as described in more detail in the following definition.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain, and fix $\{w_1, \ldots, w_N\}$, a finite set of $C(\overline{\Omega}, \mathbb{R}^K)$-functions. Define the constant M_1 by*

$$M_1 := \max_{k=1,\ldots,N} \|w_k\|_{L^\infty(\Omega)}. \tag{2.15}$$

Notice that instead of writing $\||w_k|\|_{L^\infty(\Omega)}$, where $|w_k|$ denotes the standard Euclidean norm on $\mathbb{R}^K$, we write $\|w_k\|_{L^\infty(\Omega)}$ in (2.15). This abbreviation will be used throughout the remainder of this paper.

In addition, let $c_1, \ldots, c_N$ be arbitrary real numbers, and fix nonnegative real numbers $\mu_1, \ldots, \mu_N$. For some $T > 0$ we consider functions of the form

$$w(t,x) = \sum_{k=1}^{N} e^{-\mu_k \cdot t} \cdot c_k \cdot w_k(x) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad x \in \Omega\ . \tag{2.16}$$

Finally, let $\Omega_T = [0,T] \times \Omega$ and define

$$\mathcal{M}(w) = \left\{ (t,x) \in [0,T] \times \Omega \,:\, |w(t,x)| \ge \frac{1}{2} \cdot \|w\|_{L^\infty(\Omega_T)} \right\}\ , \tag{2.17}$$

where again we use the abbreviation introduced after (2.15).

The main result of this section provides a sharp upper bound on the $L^\infty(\Omega_T)$-norm of time-dependent functions $|w|$, where w is defined in (2.16). For this, we need to make sure that the set $\mathcal{M}(w)$ defined in (2.17), which corresponds to large function values of $|w|$, cannot be too small. This is addressed in the following assumption of a uniform lower bound for $|\mathcal{M}(w)|$. Its validity will be established in Section 2.2 below.

ASSUMPTION 2.2. *In the situation of Definition 2.1, we assume that there exists a constant $\Upsilon > 0$ which depends only on the constants μ_k and the functions w_k, such that for every choice of $c_1, \ldots, c_N \in \mathbb{R}$ the function w defined in (2.16) satisfies*

$$|\mathcal{M}(w)| \geq \Upsilon \cdot |\Omega_T| \ , \tag{2.18}$$

where $|\cdot|$ denotes $(n+1)$-dimensional Lebesgue measure.

The following lemma is an extension of [37, Lemma 3.3], which in turn was based on previous work by Aurich et al. [2] and Kahane [25]. Unlike in these situations, we obtain uniform $L^\infty(\Omega_T)$-bounds for vector-valued random one-parameter families as in (2.16). This extension is crucial for extending the results of [37] to non-rectangular domains.

LEMMA 2.3. *Assume the situation of Definition 2.1 and let Υ be as in Assumption 2.2. Let $a_1, \ldots, a_N$ denote independent random variables over a probability space $(F, \mathcal{F}, \mathbb{P})$ which are all normally distributed with mean 0 and variance 1. We consider random functions of the form*

$$w(t,x,\omega) = \sum_{k=1}^{N} e^{-\mu_k \cdot t} \cdot a_k(\omega) \cdot w_k(x) \ , \tag{2.19}$$

where $(t,x) \in \Omega_T = [0,T] \times \Omega$, and $\omega \in F$. Then for any $\Upsilon^* > \Upsilon/(2K)$ we have

$$\mathbb{P}\left(\|w\|_{L^\infty(\Omega_T)} \geq M_1 \cdot \sqrt{8KN \cdot \ln(2K\Upsilon^*/\Upsilon)} \right) \leq \frac{1}{\Upsilon^*} \ . \tag{2.20}$$

Proof. Consider the real-valued component functions of w and w_k given by

$$w = \left(w^{(1)}, \ldots, w^{(K)} \right) \qquad \text{and} \qquad w_k = \left(w_k^{(1)}, \ldots, w_k^{(K)} \right) \ .$$

Since each a_k is normally distributed with mean 0 and variance 1 we obtain for every $\tau \in \mathbb{R}$ and $k = 1, \ldots, N$ that the expected value $\mathbb{E}$ of the random variable $e^{\tau a_k}$ satisfies [5]

$$\mathbb{E}\left(e^{\tau \cdot a_k}\right) = \int_F e^{\tau \cdot a_k(\omega)} d\mathbb{P}(\omega) = e^{\tau^2/2} \ .$$

Now let $\alpha \in \mathbb{R}$ be arbitrary, let $k \in \{1, \ldots, N\}$, and let $\ell \in \{1, \ldots, K\}$. Then for all $t \in [0,T]$ and $x \in \Omega$ the inequality $\mu_k \geq 0$ implies

$$\mathbb{E}\left(e^{\alpha \cdot e^{-\mu_k \cdot t} \cdot a_k \cdot w_k^{(\ell)}(x)} \right) = e^{\alpha^2 \cdot e^{-2\mu_k \cdot t} \cdot w_k^{(\ell)}(x)^2/2} \leq e^{\alpha^2 \cdot \|w_k^{(\ell)}\|^2_{L^\infty(\Omega)}/2} \leq e^{\alpha^2 \cdot M_1^2/2} \ ,$$

and the independence of the a_k's furnishes for every $\ell = 1, \ldots, K$ the estimate

$$\mathbb{E}\left(e^{\alpha \cdot w^{(\ell)}(t,x,\cdot)} \right) = \prod_{k=1}^{N} \mathbb{E}\left(e^{\alpha \cdot e^{-\mu_k \cdot t} \cdot a_k \cdot w_k^{(\ell)}(x)} \right) \leq e^{\alpha^2 \cdot M_1^2 \cdot N/2} \ ,$$

where M_1 was defined in Definition 2.1. Now assume $\alpha \ge 0$. Then for any $t \in [0,T]$, the definition in (2.17) and estimate (2.18) imply

$$\begin{aligned}
&\Upsilon \cdot |\Omega_T| \cdot \mathbb{E}\left(e^{\alpha \cdot \|w\|_{L^\infty(\Omega_T)}/2}\right) \;\le\; \mathbb{E}\int_{\mathcal{M}(w)} e^{\alpha\cdot|w(t,x,\cdot)|}\, d(t,x) \\
\le\;& \mathbb{E}\int_{\mathcal{M}(w)} e^{\alpha\cdot\sqrt{K}\cdot|w(t,x,\cdot)|_\infty}\, d(t,x) \;\le\; \sum_{\ell=1}^{K}\mathbb{E}\int_{\mathcal{M}(w)} e^{\alpha\cdot\sqrt{K}\cdot|w^{(\ell)}(t,x,\cdot)|}\, d(t,x) \\
\le\;& \sum_{\ell=1}^{K}\mathbb{E}\int_{\mathcal{M}(w)}\left(e^{\alpha\cdot\sqrt{K}\cdot w^{(\ell)}(t,x,\cdot)} + e^{-\alpha\cdot\sqrt{K}\cdot w^{(\ell)}(t,x,\cdot)}\right) d(t,x) \\
\le\;& \sum_{\ell=1}^{K}\int_{[0,T]\times\Omega}\mathbb{E}\left(e^{\alpha\cdot\sqrt{K}\cdot w^{(\ell)}(t,x,\cdot)} + e^{-\alpha\cdot\sqrt{K}\cdot w^{(\ell)}(t,x,\cdot)}\right) d(t,x) \\
\le\;& 2KT\cdot|\Omega|\cdot e^{\alpha^2\cdot K\cdot M_1^2\cdot N/2}\,,
\end{aligned}$$

where $|w(t,x,\omega)|_\infty = \max\{|w^{(1)}(t,x,\omega)|,\ldots,|w^{(K)}(t,x,\omega)|\}$. This furnishes the inequality

$$\mathbb{E}\left(e^{\alpha\cdot\|w\|_{L^\infty(\Omega_T)}/2}\right) \le \frac{2K}{\Upsilon}\cdot e^{\alpha^2\cdot M_1^2\cdot KN/2}\,.$$

Notice that in the above estimates, both the set $\mathcal{M}(w)$ and the norm $\|w\|_{L^\infty(\Omega_T)}$ depend on ω. For any $\Upsilon^* > \Upsilon/(2K)$ the last inequality can be rewritten as

$$\mathbb{E}\left(\exp\left(\frac{\alpha}{2}\cdot\left(\|w\|_{L^\infty(\Omega_T)} - \alpha\cdot M_1^2\cdot KN - \frac{2}{\alpha}\cdot\ln(2K\Upsilon^*/\Upsilon)\right)\right)\right) \le \frac{1}{\Upsilon^*}\,.$$

Since for any random variable ξ and any $\alpha \ge 0$ one has

$$\mathbb{P}(\xi \ge 0) \;\le\; \int_{\{\xi\ge 0\}} e^{\alpha\cdot\xi/2} d\mathbb{P} \;\le\; \mathbb{E}\left(e^{\alpha\cdot\xi/2}\right)\,,$$

we obtain

$$\mathbb{P}\left(\|w\|_{L^\infty(\Omega_T)} \ge \alpha\cdot M_1^2\cdot KN + \frac{2}{\alpha}\cdot\ln(2K\Upsilon^*/\Upsilon)\right) \le \frac{1}{\Upsilon^*}\,.$$

Choosing $\alpha = (2\cdot\ln(2K\Upsilon^*/\Upsilon)/(M_1^2KN))^{1/2}$ this finally yields (2.20), and the proof of the lemma is complete. □

As in [37], one can combine Lemma 2.3 with the weak law of large numbers to relate the maximum norm of the function $|w|$ to the standard Euclidean norm of the coefficient vector $(a_1,\ldots,a_N)$. This yields the following result.

Proposition 2.4. *Assume the situation of Definition 2.1, let Υ be as in Assumption 2.2, let $\Upsilon^* > \Upsilon/(2K)$, and let $a_1,\ldots,a_N$ be independent random variables over a common probability space $(F,\mathcal{F},\mathbb{P})$ which are all normally distributed with mean* 0 *and variance* 1. *We consider again random functions of the form (2.19). Then there exists a set $F_0 \in \mathcal{F}$ with*

$$\mathbb{P}(F_0) \ge 1 - \frac{1}{\Upsilon^*} - \frac{8}{N}$$

such that for every $\omega \in F_0$ and all $t \in [0,T]$ we have

$$\|w(t,\cdot,\omega)\|_{L^\infty(\Omega)} \le 4M_1\cdot\sqrt{K\cdot\ln(2K\Upsilon^*/\Upsilon)}\cdot|a(\omega)|\,, \tag{2.21}$$

where $a(\omega) = (a_1(\omega),\ldots,a_N(\omega)) \in \mathbb{R}^N$.

Proof. Since the random variables a_n are independent and normally distributed with mean 0 and variance 1, one easily obtains $\mathbb{E}(|a|^2) = \mathbb{E}(a_1^2 + \ldots + a_N^2) = N$. Furthermore, the variance of $|a|^2$ is given by $\mathbb{V}(|a|^2) = 2N$; see for example Bauer [5, §4]. The weak law of large numbers, in the form of Chebyshev's inequality, then implies

$$\mathbb{P}\left(\big| |a|^2 - N \big| \geq \frac{N}{2} \right) \leq \frac{4}{N^2} \cdot \mathbb{V}(|a|^2) = \frac{8}{N} \ ,$$

and therefore

$$\mathbb{P}\left(|a| \leq \sqrt{\frac{N}{2}} \right) \leq \frac{8}{N} \ , \tag{2.22}$$

Now let $F_0 \in \mathcal{F}$ consist of all those $\omega \in F$ for which both $|a(\omega)| > \sqrt{N/2}$ and $\sup_{t\in[0,T]} \|w(t,\cdot,\omega)\|_{L^\infty(\Omega)} < M_1 \cdot \sqrt{8KN \cdot \ln(2K\Upsilon^*/\Upsilon)}$ are satisfied. Then (2.22) and Lemma 2.3 imply $\mathbb{P}(F_0) \geq 1 - 1/\Upsilon^* - 8/N$, which completes the proof of the proposition. □

The following theorem constitutes our central result on the maximum norm of random sums. Instead of considering randomly distributed coefficients, we now choose coefficient vectors from a sphere in $\mathbb{R}^N$. The result implies that for most of these vectors, with respect to the uniform measure on the sphere, sharp estimates on the maximum norm can be obtained. In this form the result will be applied later to the Cahn-Hilliard equation on the disk.

THEOREM 2.5. *Assume the situation of Definition 2.1, let Υ be as in Assumption 2.2, and for $R > 0$ let $S_R = \left\{ a \in \mathbb{R}^N \ : \ |a| = R \right\}$ denote the sphere of radius R. Finally, let m_R denote the (unique) uniform Haar probability measure on S_R and let $\Upsilon^* > \Upsilon/(2K)$. Then there exists a measurable set $S^* \subset S_R$ with $m_R(S^*) \geq 1 - 1/\Upsilon^* - 8/N$ such that the following is true. If $a = (a_1, \ldots, a_N) \in S^*$ and if w is defined as*

$$w(t,x) = \sum_{k=1}^{N} e^{-\mu_k \cdot t} \cdot a_k \cdot w_k(x) \ ,$$

then we have

$$\|w(t,\cdot)\|_{L^\infty(\Omega)} \leq 4M_1 \cdot \sqrt{K \cdot \ln(2K\Upsilon^*/\Upsilon)} \cdot |a| \quad \text{for all} \quad t \in [0,T] \ . \tag{2.23}$$

Proof. The result is an immediate consequence of Proposition 2.4 and the following fact: If $a_1, \ldots, a_N$ are independent random variables over a common probability space $(F, \mathcal{F}, \mathbb{P})$ which are normally distributed with mean 0 and variance 1, then the mapping

$$F \ni \omega \mapsto \frac{R}{|a(\omega)|} \cdot a(\omega) \in S_R$$

maps the measure $\mathbb{P}$ to the Haar measure m_R on S_R; see for example Muirhead [30, Section 1.5]. The result now follows if we choose the set S^* as the image of the set F_0 from Proposition 2.4 under the above mapping, since (2.21) is invariant under scalings of a. □

2.2. **The Set of Large Function Values.** One of the crucial ingredients for the results of the previous section is Assumption 2.2 — and this assumption was exactly the reason for only considering rectangular domains in [37]. The assumption can easily be verified if the functions w_k are scalar-valued and form an orthonormal set, see Lemma 3.7 in [37]. Yet, in the application to the Cahn-Hilliard equation it is essential to have the results of the last section available also for first-order derivatives of the random sums. In the case of rectangular domains, these partial derivatives can still be written as linear combinations of an orthonormal set, albeit a different one. Consequently, [37, Lemma 3.7] is still applicable. However, this is no longer true for more complicated domains, most notably not even for the disk. These complications can be avoided by applying the vector-valued results of the previous section to the gradients directly. For this, we have to extend [37, Lemma 3.7] to the vector-valued case as well. First, however, we present the necessary assumptions.

ASSUMPTION 2.6. *Let Ω denote a bounded domain in $\mathbb{R}^n$, where $n \in \{1, 2, 3\}$. Moreover, let $\{w_1, \ldots, w_N\}$ be a finite set of $C^1(\overline{\Omega}, \mathbb{R}^K)$-functions, and let M_1 and M_2 be constants satisfying*

$$\|w_k\|_{L^\infty(\Omega)} \le M_1 \qquad \text{and} \qquad \left\| \frac{\partial w_k}{\partial x_\ell} \right\|_{L^\infty(\Omega)} \le M_2$$

for all $k = 1, \ldots, N$ and $\ell = 1, \ldots, n$. (Again, we employ the abbreviation introduced in Definition 2.1.) Assume that the functions w_k are orthonormal, i.e., we have

$$(w_k, w_j) = \int_\Omega (w_k(x), w_j(x))_{\mathbb{R}^K} \, dx = \delta_{k,j} \qquad \text{for} \qquad k, j = 1, \ldots, N \; .$$

Finally, suppose that the domain Ω satisfies the cone condition [1, 4.3]*: There exists a finite cone $\mathcal{C}$ such that each point $x \in \Omega$ is the vertex of a cone $\mathcal{C}_x$ contained in Ω and congruent to $\mathcal{C}$. We denote the height of $\mathcal{C}$ by $r_{\mathcal{C}}$. (This height corresponds to the radius of the ball B_1 in Adams* [1, 4.1]*.)*

LEMMA 2.7. *Suppose that Assumption 2.6 is satisfied, and let $\mu_1, \ldots, \mu_N$ be nonnegative numbers satisfying $0 \le \mu_k \le M_3$ for all $k = 1, \ldots, N$, for some constant M_3. Furthermore, let $T > 0$ be fixed, and assume that the constant r defined as*

$$r = \frac{1}{N \cdot \sqrt{|\Omega| \cdot 8K \left(M_1^2 M_3^2 + K M_2^2\right)}} \tag{2.24}$$

satisfies $r \le \min\{r_{\mathcal{C}}, T\}$. Let w be any function of the form (2.16). Then we have

$$|\mathcal{M}(w)| \ge r^{n+1} \cdot \frac{|\mathcal{C}|}{r_{\mathcal{C}}^n} \; , \tag{2.25}$$

where the set $\mathcal{M}(w)$ was defined in (2.17), and $|\cdot|$ denotes Lebesgue measure.

Proof. Due to the orthonormality of the functions w_k we have $c_k = (w(0, \cdot), w_k)$, and therefore

$$\begin{aligned} |c_k| &= |(w(0,\cdot), w_k)| \le \|w(0,\cdot)\|_{L^\infty(\Omega)} \cdot \int_\Omega |w_k| dx \\ &\le \|w\|_{L^\infty(\Omega_T)} \cdot \sqrt{|\Omega|} \cdot \|w_k\|_{L^2(\Omega)} = \sqrt{|\Omega|} \cdot \|w\|_{L^\infty(\Omega_T)} \; . \end{aligned}$$

For $(t, x) \in [0, T] \times \Omega$ this estimate implies

$$\left| \frac{\partial w}{\partial t}(t, x) \right| \le \sum_{k=1}^N \mu_k \cdot e^{-\mu_k \cdot t} \cdot |c_k| \cdot |w_k(x)| \le N \cdot M_1 M_3 \cdot \sqrt{|\Omega|} \cdot \|w\|_{L^\infty(\Omega_T)} \; ,$$

as well as

$$\left|\frac{\partial w}{\partial x_\ell}(t,x)\right| \le \sum_{k=1}^{N} e^{-\mu_k \cdot t} \cdot |c_k| \cdot \left|\frac{\partial w_k}{\partial x_\ell}(x)\right| \le N \cdot M_2 \cdot \sqrt{|\Omega|} \cdot \|w\|_{L^\infty(\Omega_T)} \; ,$$

for all $\ell = 1, \ldots, n$. Thus, if we denote the component functions of the superposition w as $w = (w^{(1)}, \ldots, w^{(K)})$, we obtain the gradient estimates

$$\left|\nabla_{(t,x)} w^{(m)}(t,x)\right| \le N \cdot \sqrt{|\Omega| \cdot (M_1^2 M_3^2 + K M_2^2)} \cdot \|w\|_{L^\infty(\Omega_T)} \; .$$

Now let $(t_*, x_*) \in [0,T] \times \overline{\Omega}$ be such that $|w(t_*, x_*)| = \|w\|_{L^\infty(\Omega_T)}$. Fix an interval $I_* \subset [0,T]$ of length r which contains t_*, and let $\mathcal{C}_* \subset \Omega$ denote a cone with vertex x_* which is congruent to $(r/r_{\mathcal{C}}) \cdot \mathcal{C}$. Then for all $(t,x) \in I_* \times \mathcal{C}_*$ we have

$$\begin{aligned} |w(t,x) - w(t_*, x_*)| &\le \sqrt{K} \cdot \max_{1 \le \ell \le K} \left\|\nabla_{(t,x)} w^{(m)}\right\|_{L^\infty(\Omega_T)} \cdot \sqrt{|t - t_*|^2 + |x - x_*|^2} \\ &\le \sqrt{K} \cdot N \cdot \sqrt{|\Omega| \cdot (M_1^2 M_3^2 + K M_2^2)} \cdot \|w\|_{L^\infty(\Omega_T)} \cdot \sqrt{2} \cdot r \\ &= \frac{1}{2} \cdot \|w\|_{L^\infty(\Omega_T)} \; , \end{aligned}$$

according to the definition of r in (2.24). This implies $I_* \times \mathcal{C}_* \subset \mathcal{M}(w)$, and noting that $|I_* \times \mathcal{C}_*| = r \cdot (r/r_{\mathcal{C}})^n \cdot |\mathcal{C}|$ completes the proof of the lemma. □

2.3. **Superpositions of Eigenfunctions.** The results of Sections 2.1 and 2.2 apply to a variety of situations, under fairly weak assumptions. In our application we have more structure to work with, as is summarized in the following definition.

ASSUMPTION 2.8. *Let Ω denote a bounded domain in $\mathbb{R}^n$, where $n \in \{1,2,3\}$, and assume that the boundary $\partial\Omega$ is Lipschitz. Let $\{\bar\psi_1, \ldots, \bar\psi_N\}$ denote an $L^2(\Omega)$-orthonormal set of eigenfunctions of the negative Laplacian $-\Delta$ on Ω subject to homogeneous Neumann boundary conditions, and denote the corresponding eigenvalues by $\kappa_1, \ldots, \kappa_N$. Let E_1 and E_2 denote positive constants such that*

$$\left\|\bar\psi_k\right\|_{L^\infty(\Omega)} \le E_1 \qquad \text{and} \qquad \left\|\nabla\bar\psi_k\right\|_{L^\infty(\Omega)} \le E_2$$

for all $k = 1, \ldots, N$. Let $\mu_1, \ldots, \mu_N$ and $b_1, \ldots, b_N$ denote arbitrary real numbers satisfying

$$0 \le \mu_k \le E_3 \qquad \text{and} \qquad |b_k| \le E_4$$

for all $k = 1, \ldots, N$ and suitable constants $E_3, E_4 > 0$. Finally, assume that E_5 and E_6 are positive constants such that for $k = 1, \ldots, N$ we have

$$E_5 \le \kappa_k \le E_6 \; .$$

Then we consider superpositions of the form

$$u(t,x) = \sum_{k=1}^{N} e^{-\mu_k \cdot t} \cdot a_k \cdot b_k \cdot \bar\psi_k(x) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad x \in \Omega \; , \tag{2.26}$$

where $a_1, \ldots, a_N$ denote arbitrary real numbers. Finally, due to our above assumption the domain Ω satisfies the cone condition [1]. *We denote the corresponding cone as in Assumption 2.6 by $\mathcal{C}$, and its height by $r_{\mathcal{C}}$.*

At first sight the introduction of two sets of coefficients a_k and b_k might seem unnecessary. However, in the Cahn-Hilliard setting the natural phase space is given by the Sobolev space $H^2(\Omega)$, rather than $L^2(\Omega)$. In this situation, we will choose the coefficients b_k in such a way that $\{b_1\bar\psi_1, \ldots, b_N\bar\psi_N\}$ denotes an orthonormal set in the phase space — and therefore the Sobolev norm of $u(0,\cdot)$ will be given by the Euclidean norm of the vector $a = (a_1, \ldots, a_N)$.

In order to apply the results on Section 2.1 to functions of the form (2.26) as well as their gradients, we first have to determine the size of the set of large function values for both u and ∇u. For this, we define the sets

$$\mathcal{M}(u) = \left\{ (t,x) \in \Omega_T \, : \, |u(t,x)| \ge \|u\|_{L^\infty(\Omega_T)}/2 \right\} , \qquad (2.27)$$

$$\mathcal{M}(\nabla u) = \left\{ (t,x) \in \Omega_T \, : \, |\nabla u(t,x)| \ge \|\nabla u\|_{L^\infty(\Omega_T)}/2 \right\} \qquad (2.28)$$

as before, with $\Omega_T = [0,T] \times \Omega$. The following lemma shows how Lemma 2.7 can be applied in this situation.

LEMMA 2.9. *Suppose that Assumption 2.8 is satisfied. Furthermore, let $T > 0$ be fixed, assume that the constants r_0 and r_1 are defined as*

$$r_0 = \frac{1}{2\sqrt{2}N \cdot |\Omega|^{1/2} \cdot \left(E_1^2 E_3^2 + E_2^2\right)^{1/2}} \qquad (2.29)$$

$$r_1 = \frac{E_5^{1/2}}{2\sqrt{2}N \cdot |\Omega|^{1/2} \cdot n^{1/2} \cdot \left(E_2^2 E_3^2 + 4nc^2\left(1 + E_6^4\right)\right)^{1/2}} , \qquad (2.30)$$

and that $\max\{r_0, r_1\} \le \min\{r_{\mathcal{C}}, T\}$. (The constant c appearing in the definition of r_1 is introduced in the proof below and depends only on the domain Ω.) If u denotes any function of the form (2.26), then we have

$$|\mathcal{M}(u)| \ge r_0^{n+1} \cdot \frac{|\mathcal{C}|}{r_{\mathcal{C}}^n} \qquad \text{and} \qquad |\mathcal{M}(\nabla u)| \ge r_1^{n+1} \cdot \frac{|\mathcal{C}|}{r_{\mathcal{C}}^n} ,$$

where the sets $\mathcal{M}(u)$ and $\mathcal{M}(\nabla u)$ were defined in (2.27) and (2.28).

Proof. The lower bound on the Lebesgue measure of $\mathcal{M}(u)$ is an immediate consequence of Lemma 2.7. We just have to set $c_k = a_k \cdot b_k$ in (2.16), as well as $w_k = \bar{\psi}_k$, $M_\eta = E_\eta$ for $\eta = 1,2,3$, and $K = 1$ in the situation of Lemma 2.7.

We now turn our attention to the gradient estimate. Let $\bar{\psi}_k$ and $\bar{\psi}_j$ denote two of the basis functions in Assumption 2.8. Since both are eigenfunctions of $-\Delta$ subject to homogeneous Neumann boundary conditions, integration by parts furnishes

$$(\nabla\bar{\psi}_k, \nabla\bar{\psi}_j) = \int_\Omega (\nabla\bar{\psi}_k(x), \nabla\bar{\psi}_j(x))_{\mathbb{R}^n} \, dx = -\int_\Omega \bar{\psi}_k(x) \Delta\bar{\psi}_j(x) \, dx = \kappa_j \cdot \delta_{k,j} .$$

Thus, if we define $w_k = \kappa_k^{-1/2} \cdot \nabla\bar{\psi}_k$, then the vector-valued functions w_k are orthonormal in the sense of Assumption 2.6, and Assumption 2.8 furnishes

$$\|w_k\|_{L^\infty(\Omega)} \le E_2 \cdot E_5^{-1/2} . \qquad (2.31)$$

We now have to establish an upper bound on the maximum norm of the first-order derivatives of the functions w_k, i.e., on the second-order derivatives of the $\bar{\psi}_k$. Due to the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, there exists a constant $c > 0$ which depends only on the domain Ω such that for all $\ell, m = 1, \ldots, n$ we have

$$\begin{aligned} \left\| \frac{\partial^2 \bar{\psi}_k}{\partial x_\ell \partial x_m} \right\|_{L^\infty(\Omega)} &\le c \cdot \left\| \frac{\partial^2 \bar{\psi}_k}{\partial x_\ell \partial x_m} \right\|_{H^2(\Omega)} \le c \cdot \left\| \bar{\psi}_k \right\|_{H^4(\Omega)} \\ &\le c \cdot \left(\left\| \Delta^2 \bar{\psi}_k \right\|_{L^2(\Omega)}^2 + \left\| \bar{\psi}_k \right\|_{L^2(\Omega)}^2 \right)^{1/2} , \end{aligned}$$

after possibly redefining c, where for the last inequality we used [35, Lemma III.4.2]. Notice that the constant c still only depends on the domain Ω. Since $\bar{\psi}_k$ is a normalized eigenfunction of $-\Delta$ we now obtain

$$\left\| \frac{\partial \nabla \bar{\psi}_k}{\partial x_\ell} \right\|_{L^\infty(\Omega)} \le n^{1/2} \cdot \max_{m=1,\ldots,n} \left\| \frac{\partial^2 \bar{\psi}_k}{\partial x_\ell \partial x_m} \right\|_{L^\infty(\Omega)} \le c \cdot n^{1/2} \cdot \left(\kappa_k^4 + 1 \right)^{1/2} ,$$

as well as

$$\left\| \frac{\partial w_k}{\partial x_\ell} \right\|_{L^\infty(\Omega)} \le 2c \cdot \left(\frac{\kappa_k^4 + 1}{\kappa_k} \right)^{1/2} \le 2c \cdot \left(E_6^4 + 1 \right)^{1/2} \cdot E_5^{-1/2} . \qquad (2.32)$$

If we finally denote the right-hand sides of (2.31) and (2.32) by M_1 and M_2, respectively, then we can apply Lemma 2.7 with $c_k = a_k \cdot b_k \cdot \kappa_k^{1/2}$ in (2.16), as well as $K = n$. This completes the proof of the lemma. □

The above lemma clears the way for applying the results of Section 2.1 to superpositions of Laplacian eigenfunctions of the form (2.26), as well as their gradients.

THEOREM 2.10. *Suppose that Assumption 2.8 holds, let r_0 and r_1 be defined as in (2.29) and (2.30), respectively, and let*

$$\Upsilon_0 = \frac{r_0^{n+1} \cdot |\mathcal{C}|}{r_{\mathcal{C}}^n \cdot T \cdot |\Omega|} \qquad \text{and} \qquad \Upsilon_1 = \frac{r_1^{n+1} \cdot |\mathcal{C}|}{r_{\mathcal{C}}^n \cdot T \cdot |\Omega|} \qquad (2.33)$$

For $R > 0$ let $S_R = \left\{ a \in \mathbb{R}^N \ : \ |a| = R \right\}$ denote the sphere of radius R, and let m_R denote the uniform Haar measure on S_R. Finally, let $\Upsilon^* > \max\{\Upsilon_0/2, \Upsilon_1/(2n)\}$.

Then there exists a measurable set $S^* \subset S_R$ with $m_R(S^*) \ge 1 - 2/\Upsilon^* - 16/N$ such that the following is true. If $a = (a_1, \ldots, a_N) \in S^*$ and if u is defined as

$$u(t,x) = \sum_{k=1}^{N} e^{-\mu_k \cdot t} \cdot a_k \cdot b_k \cdot \bar{\psi}_k(x) ,$$

then we have both

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \le 4E_1 E_4 \cdot \sqrt{\ln(2\Upsilon^*/\Upsilon_0)} \cdot |a| \qquad \text{and} \qquad (2.34)$$

$$\|\nabla u(t,\cdot)\|_{L^\infty(\Omega)} \le 4E_2 E_4 \cdot \sqrt{n \ln(2n\Upsilon^*/\Upsilon_1)} \cdot |a| \qquad (2.35)$$

for all $t \in [0,T]$.

Proof. For the proof, we only have to apply our abstract Theorem 2.5 twice. To begin with, set $w_k = b_k \cdot \bar{\psi}_k$ and $K = 1$. Then due to Assumption 2.8 the constant M_1 in Definition 2.1 can be chosen as $M_1 = E_1 \cdot E_4$, and the constant Υ in Assumption 2.2 is given by Υ_0 defined in the formulation of the theorem. Now Theorem 2.5 furnishes a measurable set $S_0^* \subset S_R$ with $m_R(S_0^*) \ge 1 - 1/\Upsilon^* - 8/N$ such that (2.34) holds for all $a \in S_0^*$.

In order to establish (2.35) we apply Theorem 2.5 one more time by considering the vector-valued case $K = n$ with $w_k = b_k \cdot \nabla \bar{\psi}_k$. This time, the constant M_1 in Definition 2.1 can be chosen as $M_1 = E_2 \cdot E_4$, and the constant Υ in Assumption 2.2 is given by the constant Υ_1 defined above. An application of Theorem 2.5 yields a measurable set $S_1^* \subset S_R$ with $m_R(S_1^*) \ge 1 - 1/\Upsilon^* - 8/N$ such that (2.35) holds for all $a \in S_1^*$. If we finally set $S^* = S_0^* \cap S_1^*$ then the theorem follows. □

The results of this section provide an upper bound on the L^∞-norm of both u and ∇u completely in terms of the constants introduced in Assumption 2.8. We will see later that in the application to the Cahn-Hilliard equation all of these constants

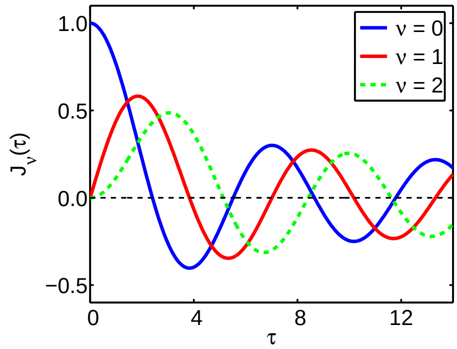

FIGURE 2. The Bessel function J_0 satisfies $J_0(0) = 1$ and is shown as solid curve. For $\nu \geq 1$ we have $J_\nu(0) = 0$, and J_1 and J_2 are shown as solid and dashed lines, respectively.

can be described in terms of the small parameter ε in (1.1), once the constants E_1 and E_2 are known. For the case of the unit disk, these latter constants will be determined in the next section.

3. **The Laplacian on the Disk.** The abstract results of the last section generalized the corresponding results of [37] to general domains. They also isolated the last crucial piece of information that has to be derived separately for each domain, namely the constants in Assumption 2.8. In order to apply Theorem 2.10 we need to establish upper bounds on the maximum norms of $L^2(\Omega)$-orthonormalized eigenfunctions of the Laplacian, as well as on the gradients of the eigenfunctions. For the case of rectangular domains, these estimates are trivial. The corresponding estimates for the case of the unit disk are substantially different, and their derivation is the subject of the current section. Throughout this section we will describe functions defined on the unit disk $D \subset \mathbb{R}^2$ in terms of polar coordinates (r, θ) with $r \in [0, 1)$ and $\theta \in [0, 2\pi)$.

We consider the eigenfunctions and eigenvalues of the negative Laplacian on the unit disk D, i.e., solutions of the problem

$$-\Delta\psi = \kappa\psi \quad \text{in} \quad D\,, \qquad \frac{\partial\psi}{\partial\nu} = 0 \quad \text{on} \quad \partial D\,. \tag{3.36}$$

To this end, consider the Bessel functions J_ν for integers $\nu \geq 0$ defined by

$$J_\nu(\tau) = \frac{\tau^\nu}{2^\nu \nu!} \cdot \left(1 - \frac{\tau^2}{2 \cdot (2\nu+2)} + \frac{\tau^4}{2 \cdot 4 \cdot (2\nu+2) \cdot (2\nu+4)} - + \ldots\right)\,,$$

which satisfy Bessel's differential equation

$$\frac{d^2 z}{d\tau^2} + \frac{1}{\tau} \cdot \frac{dz}{d\tau} + \left(1 - \frac{\nu^2}{\tau^2}\right) \cdot z = 0\,. \tag{3.37}$$

The graphs of the Bessel functions J_0, J_1, and J_2 can be found in Figure 2.

Concerning the eigenvalues and eigenfunctions of the negative Laplacian on the unit disk, the following result is well-known, see for example [12, Section V.5.5], as well as [38].

LEMMA 3.1. *For arbitrary integers $\nu \geq 0$ and $k \geq 1$ let $\sigma_{\nu,k} > 0$ denote the location of the k-th extreme value of the Bessel function J_ν in $(0, \infty)$, and let*

$$\kappa_{\nu,k} := \sigma_{\nu,k}^2 \quad \text{for} \quad \nu \in \mathbb{N}_0 \; , \quad k \in \mathbb{N} \; .$$

Furthermore, for $r \in [0, 1)$ and $\theta \in [0, 2\pi]$ define functions

$$\begin{aligned} \psi_{0,k}(r, \theta) &:= a_{0,k} \cdot J_0 \left(\sigma_{0,k} \cdot r\right) \; , \\ \psi_{\nu,k,c}(r, \theta) &:= a_{\nu,k} \cdot J_\nu \left(\sigma_{\nu,k} \cdot r\right) \cdot \cos(\nu\theta) \; , \\ \psi_{\nu,k,s}(r, \theta) &:= a_{\nu,k} \cdot J_\nu \left(\sigma_{\nu,k} \cdot r\right) \cdot \sin(\nu\theta) \; , \end{aligned} \tag{3.38}$$

for $\nu, k \in \mathbb{N}$, with suitable scaling factors $a_{\nu,k} > 0$ which will be specified below.

Then the eigenvalues of (3.36) are given by 0 and the numbers $\kappa_{\nu,k}$ for $\nu \in \mathbb{N}_0$ and $k \in \mathbb{N}$. The eigenvalue 0 is simple, and the corresponding eigenfunction is constant. For $k \in \mathbb{N}$ the eigenvalues $\kappa_{0,k}$ are also simple, with corresponding eigenfunctions $\psi_{0,k}$. For $\nu > 0$ and $k \in \mathbb{N}$ the eigenspace corresponding to the eigenvalue $\kappa_{\nu,k}$ is two-dimensional and spanned by the eigenfunctions $\psi_{\nu,k,c}$ and $\psi_{\nu,k,s}$.

In order to simplify the notation, we will sometimes write $\psi_{0,k,c}$ instead of $\psi_{0,k}$ in the following.

The above lemma shows that the functions defined in (3.38) form a complete orthogonal set in the Hilbert space $X \subset L^2(D)$ defined in (1.3). For our purposes, it is important to also normalize them. This is the subject of the following lemma.

LEMMA 3.2. *In the situation of Lemma 3.1, define the scaling factors $a_{\nu,k}$ as*

$$\begin{aligned} a_{0,k} &:= \frac{\pi^{-1/2}}{|J_0(\sigma_{0,k})|} \; , \\ a_{\nu,k} &:= \frac{2^{1/2} \cdot \pi^{-1/2}}{(1 - \nu^2/\kappa_{\nu,k})^{1/2} \cdot |J_\nu(\sigma_{\nu,k})|} \qquad \text{for} \qquad \nu > 0 \; . \end{aligned}$$

Then the functions defined in (3.38) form a complete orthonormal set in $X \subset L^2(D)$.

Proof. We only consider the eigenfunctions $\psi_{\nu,k,s}$ for $\nu, k \in \mathbb{N}$, since the remaining assertions can be shown similarly. Due to

$$\|\psi_{\nu,k,s}\|_{L^2(D)}^2 = a_{\nu,k}^2 \cdot \int_0^{2\pi} \sin^2(\nu\theta) \, d\theta \cdot \int_0^1 J_\nu^2(\sigma_{\nu,k} r) r \, dr = \frac{\pi a_{\nu,k}^2}{\sigma_{\nu,k}^2} \cdot \int_0^{\sigma_{\nu,k}} J_\nu^2(\tau)\tau \, d\tau,$$

we only have to calculate the last integral. For this, recall that J_ν solves the ordinary differential equation (3.37). Multiplying this equation by $\tau^2 \cdot dz/d\tau$ one obtains

$$\left(\tau \cdot \frac{dz}{d\tau}\right) \cdot \frac{d}{d\tau} \left(\tau \cdot \frac{dz}{d\tau}\right) + \left(\tau^2 - \nu^2\right) \cdot z \cdot \frac{dz}{d\tau} = 0 \; ,$$

and integration from $\tau = 0$ to $\tau = \sigma_{\nu,k}$ furnishes after integration by parts the identity

$$\int_0^{\sigma_{\nu,k}} J_\nu^2(\tau)\tau \, d\tau = \frac{\sigma_{\nu,k}^2}{2} \cdot J_\nu'(\sigma_{\nu,k})^2 + \frac{\sigma_{\nu,k}^2 - \nu^2}{2} \cdot J_\nu(\sigma_{\nu,k})^2 + \frac{\nu^2}{2} \cdot J_\nu(0)^2 \; .$$

Together with $J_\nu'(\sigma_{\nu,k}) = J_\nu(0) = 0$ this completes the proof of the lemma. □

It is clear from our abstract results in Section 2 that we are particularly interested in estimates on the $L^\infty(D)$-norm of the orthonormal eigensystem introduced in

Lemma 3.2. This basically amounts to understanding the asymptotic behavior of $J_\nu(\sigma_{\nu,k})$ as $\sigma_{\nu,k} \to \infty$. For fixed $\nu \in \mathbb{N}_0$ we have the asymptotic representation

$$J_\nu(\tau) = \sqrt{\frac{2}{\pi\tau}} \cdot \cos\left(\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\tau^{-3/2}\right) \quad \text{for} \quad \tau \to \infty \,, \tag{3.39}$$

see for example [12, Section VII.6.2]. From this representation one can easily deduce that

$$a_{\nu,k} \sim \sqrt{\sigma_{\nu,k}} \quad \text{for fixed } \nu \text{ and } k \to \infty \,,$$

and in combination with standard properties of the Bessel functions this immediately implies that the $L^\infty(D)$-norm of the eigenfunctions $\psi_{\nu,k,c/s}$ is at least of the order $\kappa_{\nu,k}^{1/4}$. In fact, we have the following result.

Lemma 3.3. *Consider the eigenfunctions of the problem (3.36) as in Lemma 3.1, with scaling factors as in Lemma 3.2. Then there exists a constant $C > 0$ such that for all $\nu \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have*

$$\left\|\psi_{\nu,k,c/s}\right\|_{L^\infty(D)} \le C \cdot \kappa_{\nu,k}^{1/4} \,. \tag{3.40}$$

This estimate is sharp and cannot be improved.

Proof. The estimate in (3.40) is a special case of [21, Theorem 1], which contains an L^∞-estimate for eigenfunctions of arbitrary Riemannian manifolds with boundary. A direct proof using asymptotic expansions of the Bessel functions similar to (3.39), but uniform in ν and τ, can be found in [20]. The optimality of (3.40) has already been established above using (3.39). □

In addition to the bound on the eigenfunctions, we also need to determine the asymptotics of the $L^\infty(D)$-norm of the gradients of the eigenfunctions. This is the subject of the following lemma.

Lemma 3.4. *Consider the eigenfunctions of the problem (3.36) as in Lemma 3.1, with scaling factors as in Lemma 3.2. Then there exists a constant $C > 0$ such that for all $\nu \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have*

$$\left\|\nabla\psi_{\nu,k,c/s}\right\|_{L^\infty(D)} \le C \cdot \kappa_{\nu,k}^{3/4} \,. \tag{3.41}$$

Also this estimate is sharp and cannot be improved. As in Section 2, we use the abbreviation introduced in Definition 2.1 after (2.15).

Proof. We begin by establishing an auxiliary estimate relating the maximum norm of the derivative of a Bessel function to the maximum norm of the Bessel function. Due to [38, Equation 2.12(4)] the derivatives of the Bessel functions satisfy the identity

$$J_\nu'(\tau) = \frac{\nu}{\tau} \cdot J_\nu(\tau) - J_{\nu+1}(\tau) \tag{3.42}$$

for all $\tau > 0$, $\nu \in \mathbb{N}$. A direct check shows that this formula also holds for $\nu = 0$. Following the notation in [38], we denote the smallest positive zero of J_ν'' by j_ν''. Then according to [38, Estimates 15.3(6)] we have

$$\sqrt{\nu(\nu-1)} < j_\nu'' < \sqrt{\nu^2 - 1}$$

for all $\nu \ge 2$. If $\tau_\nu > 0$ is chosen such that $|J_\nu'(\tau_\nu)| = \|J_\nu'\|_{L^\infty(\mathbb{R}^+)}$, then the above statements, combined with the fact that $\|J_\nu\|_{L^\infty(\mathbb{R}^+)}$ is a decreasing function of the order ν [26, p. 202], readily furnish

$$\begin{aligned}\|J_\nu'\|_{L^\infty(\mathbb{R}^+)} &\le \frac{\nu}{\tau_\nu}\cdot|J_\nu(\tau_\nu)| + |J_{\nu+1}(\tau_\nu)| \le \frac{\nu}{j_\nu''}\cdot\|J_\nu\|_{L^\infty(\mathbb{R}^+)} + \|J_{\nu+1}\|_{L^\infty(\mathbb{R}^+)} \\ &\le \left(\frac{\nu}{\sqrt{\nu(\nu-1)}}+1\right)\cdot\|J_\nu\|_{L^\infty(\mathbb{R}^+)} \le \left(\sqrt{2}+1\right)\cdot\|J_\nu\|_{L^\infty(\mathbb{R}^+)}\end{aligned}$$

for all $\nu \ge 2$. For $\nu = 0$ the identity $J_0' = -J_1$, together with the above-mentioned monotonicity of $\|J_\nu\|_{L^\infty(\mathbb{R}^+)}$, yields

$$\|J_0'\|_{L^\infty(\mathbb{R}^+)} = \|J_1\|_{L^\infty(\mathbb{R}^+)} \le \|J_0\|_{L^\infty(\mathbb{R}^+)} .$$

For $\nu = 1$ standard properties of the Bessel functions [38] imply

$$\|J_1'\|_{L^\infty(\mathbb{R}^+)} = J_1'(0) = \frac{1}{2} < \|J_1\|_{L^\infty(\mathbb{R}^+)} \approx 0.58 .$$

Thus, we have shown that for arbitrary $\nu \in \mathbb{N}_0$ the estimate

$$\|J_\nu'\|_{L^\infty(\mathbb{R}^+)} \le \left(\sqrt{2}+1\right)\cdot\|J_\nu\|_{L^\infty(\mathbb{R}^+)} \tag{3.43}$$

holds.

After these preliminary considerations we now establish (3.41). As in Lemma 3.3 we only prove the result for the eigenfunctions $\psi_{\nu,k,s}$ for $\nu, k \in \mathbb{N}$. Using the transformation rule between cartesian and polar coordinates together with (3.38) and (3.42), one easily obtains

$$\begin{aligned}\frac{\partial\psi_{\nu,k,s}}{\partial x} &= -a_{\nu,k}\cdot\nu\cdot\sin\theta\cdot\cos(\nu\theta)\cdot\frac{1}{r}\cdot J_\nu(\sigma_{\nu,k}r) \\ &\quad + a_{\nu,k}\cdot\sigma_{\nu,k}\cdot\cos\theta\cdot\sin(\nu\theta)\cdot J_\nu'(\sigma_{\nu,k}r) \\ &= -a_{\nu,k}\cdot\nu\cdot\sin\theta\cdot\cos(\nu\theta)\cdot\frac{\sigma_{\nu,k}}{\nu}\cdot\left(J_\nu'(\sigma_{\nu,k}r) + J_{\nu+1}(\sigma_{\nu,k}r)\right) \\ &\quad + a_{\nu,k}\cdot\sigma_{\nu,k}\cdot\cos\theta\cdot\sin(\nu\theta)\cdot J_\nu'(\sigma_{\nu,k}r) .\end{aligned}$$

The monotonicity of $\|J_\nu\|_{L^\infty(\mathbb{R}^+)}$ and (3.43) now furnish

$$\begin{aligned}\left\|\frac{\partial\psi_{\nu,k,s}}{\partial x}\right\|_{L^\infty(D)} &\le a_{\nu,k}\cdot\sigma_{\nu,k}\cdot\left(2\|J_\nu'\|_{L^\infty(\mathbb{R}^+)} + \|J_{\nu+1}\|_{L^\infty(\mathbb{R}^+)}\right) \\ &\le a_{\nu,k}\cdot\sigma_{\nu,k}\cdot\left(2\sqrt{2}+3\right)\cdot\|J_\nu\|_{L^\infty(\mathbb{R}^+)} .\end{aligned}$$

If j_ν' denotes the smallest positive zero of J_ν', then it was shown in [38, Section 15.31] that $\|J_\nu\|_{L^\infty(\mathbb{R}^+)} = J_\nu(j_\nu')$, since $\nu \ge 1$. Together with $j_\nu' \le \sigma_{\nu,k}$ we now obtain

$$\|J_\nu\|_{L^\infty(\mathbb{R}^+)} = \sup_{0\le\tau\le\sigma_{\nu,k}} |J_\nu(\tau)| = \frac{1}{a_{\nu,k}}\cdot\|\psi_{\nu,k,s}\|_{L^\infty(D)} ,$$

and therefore

$$\left\|\frac{\partial\psi_{\nu,k,s}}{\partial x}\right\|_{L^\infty(D)} \le \left(2\sqrt{2}+3\right)\cdot\sigma_{\nu,k}\cdot\|\psi_{\nu,k,s}\|_{L^\infty(D)} .$$

Analogously we can bound the maximum norm of $\partial\psi_{\nu,k,s}/\partial y$, and an application of Lemmas 3.1 and 3.3 completes the proof of (3.41). In order to show that the estimate is sharp, one can again employ (3.39). □

The above results establish the precise asymptotics of the maximum of both the eigenfunctions and their gradients on the unit disk. Unlike in the case of rectangular domains, where uniform bounds are observed, these norms grow with increasing wave number. In fact, the behavior of the eigenfunctions on the disk constitutes the maximum possible growth rate for two-dimensional domains, as was shown in [20].

4. **Transient Pattern Formation.** In the last two sections we have presented all the results which are necessary for studying the formation of transient patterns on the unit disk. These results will be combined in the following. After describing the basic functional-analytic setting in Section 4.1, the crucial nonlinearity estimate is derived in Section 4.2. The main result of the paper is then presented in Section 4.3. Throughout this section, the constant C describes a positive constant which is independent of the small parameter ε in (1.1), and whose value can change from line to line.

4.1. **Functional-Analytic Setting.** We consider the Cahn-Hilliard equation as defined in (1.1), and assume the following:

(A1) Suppose that D denotes the unit disk in $\mathbb{R}^2$, and that $f : \mathbb{R} \to \mathbb{R}$ is a C^4-function. Furthermore, assume that the total mass $\mu \in \mathbb{R}$ is chosen in such a way that $f'(\mu) > 0$.

For the sake of simplicity, we follow [29, 34, 37] and perform a change of variables so that the mass constraint $\int_D u \, dx = \mu$ can be replaced by $\int_D u \, dx = 0$. This leads to the transformed equation

$$\begin{aligned} u_t &= -\Delta(\varepsilon^2 \Delta u + f(\mu + u)) \quad \text{in} \quad D , \\ \frac{\partial u}{\partial \nu} &= \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \partial D , \\ \int_D u \, dx &= 0 . \end{aligned} \tag{4.44}$$

If u is any solution of (4.44), then $\mu + u$ solves the original Cahn-Hilliard equation (1.1), and vice versa. As demonstrated in [29], the equation (4.44) can be viewed as an abstract evolution equation of the form

$$u_t = A_\varepsilon u + F(u) . \tag{4.45}$$

The linear operator A_ε is defined as in (1.2), and the nonlinearity F is given by

$$F(u) = -\Delta g(u) \quad \text{where} \quad g(u) = f(\mu + u) - f'(\mu)u - f(\mu) . \tag{4.46}$$

In order to derive our nonlinearity estimates for F in Section 4.2, we need to specify the local behavior of g.

(A2) Assume that (A1) holds and let g be defined as in (4.46). We assume that g satisfies $g(u) = u^{1+\sigma} \cdot \tilde{g}(u)$, where $\sigma \geq 1$ and $\tilde{g}$ is C^2 on an open interval containing 0.

For the standard Cahn-Hilliard equation with $\mu = 0$ and $f(u) = u - u^3$ this condition is satisfied with $\sigma = 2$, for the case $\mu \neq 0$ we have $\sigma = 1$. In the latter case we have to consider the transformed equation (4.44).

The functional-analytic setting for the abstract evolution equation (4.45) is as follows. Let X be defined as in (1.3). Then the operator $A_\varepsilon : X \to X$ with domain (1.4) is self-adjoint, and $-A_\varepsilon$ is sectorial. Thus, we can define the fractional power spaces $X^{1/2,\varepsilon} \subset X$ equipped with the operator norm $\|\cdot\|_{1/2,\varepsilon}$; see for example

Henry [22]. These spaces are Hilbert spaces, and at least in principle could depend on the parameter ε. However, it was shown in [29] that these spaces are independent of ε and coincide with the space $X^{1/2}$ defined in (1.7). While the norms $\|\cdot\|_{1/2,\varepsilon}$ do depend on ε, they are all equivalent to the standard $H^2(D)$-norm on $X^{1/2}$. For our applications, it is more convenient to work with the non-standard norm

$$\|u\|_* = \sqrt{\|u\|^2_{L^2(D)} + \|\Delta u\|^2_{L^2(D)}} \quad \text{for} \quad u \in X^{1/2} .$$

On the unit disk D this norm is equivalent to the standard $H^2(D)$-norm, and therefore also to the operator norms $\|\cdot\|_{1/2,\varepsilon}$. Furthermore, (A2) implies that the nonlinearity F defined in (4.46) is a continuously differentiable nonlinear operator from $X^{1/2}$ to X, see [19, 29]. The theory of Henry [22] now shows that the abstract evolution equation (4.45) generates a nonlinear semiflow $T_\varepsilon(t)$, $t \ge 0$, on the Hilbert space $X^{1/2}$.

The basic properties of the linearized Cahn-Hilliard operator A_ε defined in (1.2) were already mentioned in the introduction. Let $0 < \kappa_1 \le \kappa_2 \le \ldots \to \infty$ denote the eigenvalues of $-\Delta$ subject to homogeneous Neumann boundary conditions, and denote the corresponding $L^2(D)$-orthonormalized eigenfunctions by ψ_k. In other words, the sequence κ_k can be obtained by ordering the $\kappa_{\nu,k}$ from Lemma 3.1, and the eigenfunctions are obtained from the $\psi_{\nu,k,c/s}$ through this ordering procedure. Notice that for the case of two-dimensional domains, including the disk D, we have

$$\kappa_k \sim k \qquad \text{as} \qquad k \to \infty . \tag{4.47}$$

The spectrum of A_ε can easily be described using the κ_k, and it consists exactly of the eigenvalues $\lambda_{k,\varepsilon}$ defined in (1.5). As we mentioned in the introduction, the operator A_ε is the generator of an analytic semigroup $S_\varepsilon(t)$, $t \ge 0$, on the Hilbert space X. This semigroup has the explicit representation

$$S_\varepsilon(t)v = \sum_{k=1}^{\infty} e^{\lambda_{k,\varepsilon}\cdot t} \cdot (v, \psi_k)_{L^2(D)} \cdot \psi_k \quad \text{for} \quad v \in X . \tag{4.48}$$

The asymptotic behavior of $S_\varepsilon(t)$ is described in the following result, which we quote from Sander, Wanner [34], see also Wanner [37, Lemma 2.1].

LEMMA 4.1. *Let A_ε be as in (1.2), consider the spaces X and $X^{1/2}$ defined in (1.3) and (1.7), respectively, and let $S_\varepsilon(t)$ denote the analytic semigroup generated by A_ε. Let X_ε^+ be defined as in (1.8) for some $\gamma_o \in (0,1)$, and let X_ε^- denote its orthogonal complement in $X^{1/2}$. Finally, choose an arbitrary $\beta_o > 0$, and define*

$$K_\varepsilon = \frac{1}{\varepsilon} \cdot \sqrt{\frac{1 + \beta_o + 4\varepsilon^4/f'(\mu)^2}{2e \cdot \beta_o}} , \quad \beta_\varepsilon = (1+\beta_o) \cdot \lambda_\varepsilon^{\max} , \quad \text{and} \quad \gamma_\varepsilon = \gamma_o \cdot \lambda_\varepsilon^{\max} ,$$

where $\lambda_\varepsilon^{\max}$ was defined in (1.6). Then the following estimates hold:

$$\begin{aligned}
\|S_\varepsilon(t)v\|_* &\le K_\varepsilon \cdot t^{-1/2} \cdot e^{\beta_\varepsilon \cdot t} \cdot \|v\|_{L^2(D)} && \text{for} \quad t > 0 , \quad v \in X , \\
\|S_\varepsilon(t)v\|_* &\le e^{\lambda_\varepsilon^{\max} \cdot t} \cdot \|v\|_* && \text{for} \quad t \ge 0 , \quad v \in X^{1/2} , \\
\|S_\varepsilon(t)v^+\|_* &\ge e^{\gamma_\varepsilon \cdot t} \cdot \|v^+\|_* && \text{for} \quad t \ge 0 , \quad v^+ \in X_\varepsilon^+ , \\
\|S_\varepsilon(t)v^-\|_* &\le e^{\gamma_\varepsilon \cdot t} \cdot \|v^-\|_* && \text{for} \quad t \ge 0 , \quad v^- \in X_\varepsilon^- .
\end{aligned}$$

While the eigenfunctions ψ_k are orthonormalized with respect to the $L^2(D)$-norm, our main phase space is given by the Hilbert space $X^{1/2}$ with norm $\|\cdot\|_*$.

One can easily verify that if we define

$$\varphi_k = \frac{1}{\sqrt{1+\kappa_k^2}} \cdot \psi_k \quad \text{for} \quad k \in \mathbb{N} \,, \tag{4.49}$$

then the set φ_k, $k \in \mathbb{N}$, is a complete orthonormal set in this latter Hilbert space.

We close this section with a few more detailed comments on the dominating subspace X_ε^+ defined in (1.8). This definition, together with (1.5) and (1.6), implies that an eigenfunction ψ_k is contained in the dominating subspace if and only if

$$\underline{\kappa}_\varepsilon = \frac{f'(\mu)}{2\varepsilon^2} \cdot \left(1 - \sqrt{1-\gamma_o}\right) \quad \le \quad \kappa_k \quad \le \quad \overline{\kappa}_\varepsilon = \frac{f'(\mu)}{2\varepsilon^2} \cdot \left(1 + \sqrt{1-\gamma_o}\right) \,. \tag{4.50}$$

In other words, there are constants $N_{1,\varepsilon}, N_{2,\varepsilon} \in \mathbb{N}$ with $N_{1,\varepsilon} \le N_{2,\varepsilon}$ such that

$$X_\varepsilon^+ = \operatorname{span}\{\psi_k \;:\; N_{1,\varepsilon} \le k \le N_{2,\varepsilon}\} = \operatorname{span}\{\varphi_k \;:\; N_{1,\varepsilon} \le k \le N_{2,\varepsilon}\} \,. \tag{4.51}$$

Furthermore, due to (4.47) and (4.50) we have that $N_{1,\varepsilon}, N_{2,\varepsilon} \sim \varepsilon^{-2}$, and thus

$$\dim X_\varepsilon^+ \sim \varepsilon^{-2} \quad \text{for} \quad \varepsilon \to 0 \,. \tag{4.52}$$

The fact that the dimension of the dominating subspace X_ε^+ increases as $\varepsilon \to 0$ is one of the crucial factors in obtaining an accurate description of the patterns observed during spinodal decomposition. See Maier-Paape, Wanner [28].

4.2. **Nonlinearity Estimate.** It was shown in [34, 37] that the crucial step for explaining transient patterns in the Cahn-Hilliard model is the derivation of precise bounds on the nonlinearity F defined in (4.46). Such nonlinearity estimates rely on a detailed description of the temporal evolution of the L^∞-norm of certain solutions of the linearized Cahn-Hilliard equation (1.2), as well as their gradients. As we mentioned above, A_ε generates an analytic semigroup $S_\varepsilon(t)$, $t \ge 0$, on X. The unique solution of (1.2) originating at v_o is given by $v(t) = S_\varepsilon(t)v_o$, with explicit representation (4.48). We are interested in solutions of (1.2) for which v_o is contained in the dominating subspace X_ε^+. The next lemma is the key for obtaining our nonlinearity estimates.

LEMMA 4.2. *Let D denote the unit disk in $\mathbb{R}^2$, and let $T > 0$ be arbitrary. Then there exist ε-independent constants C_1 and C_2, and for every sufficiently small $\varepsilon > 0$ there exists a subset $\mathcal{G}_\varepsilon^+ \subset X_\varepsilon^+$ such that the following hold.*

(a) *For any $v^+ \in \mathcal{G}_\varepsilon^+$ and arbitrary $r > 0$ we have $r \cdot v^+ \in \mathcal{G}_\varepsilon^+$, i.e., the set $\mathcal{G}_\varepsilon^+$ is the union of half-rays originating at the origin.*

(b) *Let $r > 0$ be arbitrary, and let m_r denote the uniform probability measure on the sphere $S_{\varepsilon,r}^+ = \{v^+ \in X_\varepsilon^+ : \|v^+\|_* = r\}$. Then*

$$m_r(\mathcal{G}_\varepsilon^+ \cap S_{\varepsilon,r}^+) \ge 1 - C_1 \cdot \varepsilon^2 \,.$$

(c) *Let $v_o^+ \in \mathcal{G}_\varepsilon^+$ be arbitrary, and let $v^+(t) = S_\varepsilon(t)v_o^+$ denote the solution of the linear equation (1.2) starting at v_o^+. Then for all $0 \le t \le T$ we have*

$$\|S_\varepsilon(t)v_o^+\|_{L^\infty(D)} \quad \le \quad C_2 \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln\varepsilon|} \cdot e^{\lambda_\varepsilon^{\max} \cdot t} \cdot \|v_o^+\|_* \,, \tag{4.53}$$

$$\|\nabla S_\varepsilon(t)v_o^+\|_{L^\infty(D)} \quad \le \quad C_2 \cdot \varepsilon^{1/2} \cdot \sqrt{|\ln\varepsilon|} \cdot e^{\lambda_\varepsilon^{\max} \cdot t} \cdot \|v_o^+\|_* \,, \tag{4.54}$$

$$\|\nabla S_\varepsilon(t)v_o^+\|_{L^4(D)} \quad \le \quad C_2 \cdot \varepsilon^{3/4} \cdot \sqrt[4]{|\ln\varepsilon|} \cdot e^{\lambda_\varepsilon^{\max} \cdot t} \cdot \|v_o^+\|_* \,, \tag{4.55}$$

where again we use the notation introduced in Definition 2.1.

Proof. We only have to construct a subset $\mathcal{G}^+_{\varepsilon,1}$ of the unit sphere $S^+_{\varepsilon,1}$ in X^+_ε such that (b) is satisfied with $r = 1$ and such that (c) holds for all $v^+_o \in \mathcal{G}^+_{\varepsilon,1}$. If we then define

$$\mathcal{G}^+_\varepsilon = \bigcup_{r>0} \left(r \cdot \mathcal{G}^+_{\varepsilon,1}\right) \subset X^+_\varepsilon ,$$

then (a) is automatically satisfied, and the validity of (b) for arbitrary $r > 0$ follows easily. Furthermore, since we are considering the evolution of a linear semigroup, the estimates in (c) remain valid for arbitrary $v^+_o \in \mathcal{G}^+_\varepsilon$.

In order to construct the set $\mathcal{G}^+_{\varepsilon,1}$ we apply the results of Section 2.3, i.e., we have to determine the ε-dependence of the constants $E_1, \ldots, E_6$ in Assumption 2.8. According to the definition of the dominating subspace X^+_ε and (1.5) there exist integers $N_{1,\varepsilon} \le N_{2,\varepsilon}$ such that (4.51) holds. Now choose $\{\bar\psi_1, \ldots, \bar\psi_N\}$ in Assumption 2.8 as $\{\psi_{N_{1,\varepsilon}}, \ldots, \psi_{N_{2,\varepsilon}}\}$, where ψ_k denotes the eigenfunction of the negative Laplacian on the unit disk corresponding to the k-th eigenvalue (arranged in increasing order counting multiplicities). Due to (4.52) we obtain that $N = N_{2,\varepsilon} - N_{1,\varepsilon} + 1$ satisfies

$$N \sim \varepsilon^{-2} . \tag{4.56}$$

Applying Lemma 3.3 and Lemma 3.4, together with (4.50), furnishes

$$E_1 \sim \varepsilon^{-1/2} , \qquad E_2 \sim \varepsilon^{-3/2} , \qquad E_5 \sim \varepsilon^{-2} , \qquad \text{and} \qquad E_6 \sim \varepsilon^{-2} . \tag{4.57}$$

For $a_{N_{1,\varepsilon}}, \ldots, a_{N_{2,\varepsilon}} \in \mathbb{R}$ we consider functions of the form

$$u(t,\cdot) = e^{-\lambda^{\max}_\varepsilon \cdot t} \cdot S_\varepsilon(t) v^+_o = \sum_{k=N_{1,\varepsilon}}^{N_{2,\varepsilon}} e^{-(\lambda^{\max}_\varepsilon - \lambda_{k,\varepsilon})\cdot t} \cdot a_k \cdot b_k \cdot \psi_k ,$$

where $v^+_o = \sum_{k=N_{1,\varepsilon}}^{N_{2,\varepsilon}} a_k \cdot b_k \cdot \psi_k \in X^+_\varepsilon$ and $b_k = (1 + \kappa_k^2)^{-1/2}$. Then the scaled functions $b_k\psi_k$ are orthonormal in $X^{1/2}$ with respect to the scalar product induced by $\|\cdot\|_*$, and we have $\|v^+_o\|_* = |(a_{N_{1,\varepsilon}}, \ldots, a_{N_{2,\varepsilon}})|$. Since the factors $\lambda^{\max}_\varepsilon - \lambda_{k,\varepsilon}$ in the exponents of the exponential terms are bounded by $C \cdot \varepsilon^{-2}$, we finally obtain together with (4.50) that

$$E_3 \sim \varepsilon^{-2} \qquad \text{and} \qquad E_4 \sim \varepsilon^{2} . \tag{4.58}$$

Using the asymptotics in (4.56), (4.57), and (4.58), the constants r_0 and r_1 in (2.31) and (2.32) satisfy $r_0 \sim \varepsilon^{9/2}$ and $r_1 \sim \varepsilon^5$, respectively, and therefore we have

$$\Upsilon_0 \sim \varepsilon^{27/2} \qquad \text{and} \qquad \Upsilon_1 \sim \varepsilon^{15}$$

in (2.33) — and an application of Theorem 2.10 then furnishes (4.53) and (4.54). Thus, in order to complete the proof we only have to establish (4.55). This can be accomplished completely analogous to the proof of [37, Lemma 4.2]. □

REMARK 4.3. Notice that the set $\mathcal{G}^+_\varepsilon$ is obtained from the dominating subspace X^+_ε by removing certain cone-shaped regions. It follows easily from the above lemma that these removed parts are small in the following sense. If $|\cdot|$ denotes the canonical Lebesgue measure on the finite-dimensional space X^+_ε and if $B_R(0) \subset X^{1/2}$ denotes the ball of radius R centered at 0, then there exists an ε-independent constant $C > 0$ such that for all sufficiently small $\varepsilon > 0$ we have

$$\frac{|\mathcal{G}^+_\varepsilon \cap B_R(0)|}{|X^+_\varepsilon \cap B_R(0)|} \ge 1 - C \cdot \varepsilon^2 \quad \text{for all} \quad R > 0 .$$

Therefore, as $\varepsilon \to 0$ the sets $\mathcal{G}_\varepsilon^+$ cover as large a percentage of the dominating subspace X_ε^+ as we wish.

Lemma 4.2 is the crucial result for obtaining sharp bounds on the nonlinearity F of the Cahn-Hilliard equation. In fact, once the lemma has been established, the remaining results of [37] can be proved almost verbatim — one just has to keep track of the different exponents of ε in (4.53), (4.54), and (4.55). Thus, we arrive at the following nonlinearity estimate.

PROPOSITION 4.4. *Assume that (A1) and (A2) are satisfied, let F be as in (4.46), and let $\delta^* > 0$. Consider the dominating subspace X_ε^+ from (1.8), and let $\mathcal{G}_\varepsilon^+$ denote its subset constructed in Lemma 4.2 for $T = 1$. Then there exist constants C_3 and C_4 such that for all sufficiently small values of $\varepsilon > 0$ the following is true.*

Let $v_o^+ \in \mathcal{G}_\varepsilon^+$ be arbitrary, and let $v^+(t) = S_\varepsilon(t)v_o^+$ denote the solution of (1.2) starting at v_o^+. Furthermore, for $0 < T^* \le 1$ assume that $u : [0, T^*] \to X^{1/2}$ is any continuous function satisfying both

$$\|u(t)\|_{L^\infty(D)} \le C_3 \tag{4.59}$$

and

$$\frac{\|u(t) - v^+(t)\|_*}{\|v^+(t)\|_*} \le \delta^* \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|} \tag{4.60}$$

for all $t \in [0, T^*]$. Then for all $t \in [0, T^*]$ we have

$$\|F(u(t))\|_{L^2(D)} \le C_4 \cdot \left(\varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|}\right)^\sigma \cdot e^{(\sigma+1)\cdot\lambda_\varepsilon^{\max}\cdot t} \cdot \|v_o^+\|_*^{\sigma+1} . \tag{4.61}$$

Proof. For the sake of completeness, we sketch the proof by giving the essential steps. For more details we refer the reader to the proof of [37, Proposition 4.4].

To begin with, the Sobolev embedding $H^2(D) \hookrightarrow L^\infty(D)$, together with (4.53), (4.60), and Lemma 4.1 furnish for all $t \in [0, T^*]$ the estimate

$$\begin{aligned} \|u(t)\|_{L^\infty(D)} &\le C \cdot \|u(t) - v^+(t)\|_* + \|v^+(t)\|_{L^\infty(D)} \\ &\le C \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|} \cdot e^{\lambda_\varepsilon^{\max}\cdot t} \cdot \|v_o^+\|_* . \end{aligned} \tag{4.62}$$

The Sobolev embedding $H^2(D) \hookrightarrow W^{1,4}(D)$, in combination with (4.55), (4.60), and Lemma 4.1 yields

$$\begin{aligned} \|\nabla u(t)\|_{L^4(D)} &\le C \cdot \|u(t) - v^+(t)\|_* + \|\nabla v^+(t)\|_{L^4(D)} \\ &\le C \cdot \left(\varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|}\right)^{1/2} \cdot e^{\lambda_\varepsilon^{\max}\cdot t} \cdot \|v_o^+\|_* . \end{aligned}$$

Furthermore, (4.60) furnishes for all $t \in [0, T^*]$ and all sufficiently small $\varepsilon > 0$ the estimate

$$\|u(t)\|_* \le \|u(t) - v^+(t)\|_* + \|v^+(t)\|_* \le C \cdot e^{\lambda_\varepsilon^{\max}\cdot t} \cdot \|v_o^+\|_* .$$

The assumptions on $\tilde{g}$ furnish constants $C > 0$ and $C_3 > 0$ such that

$$|g'(s)| \le C \cdot |s|^\sigma \quad \text{and} \quad |g''(s)| \le C \cdot |s|^{\sigma-1} \quad \text{for all} \quad |s| \le C_3 ,$$

and (4.59) implies the validity of both

$$\|g'(u(t))\|_{L^\infty(D)} \le C \cdot \|u(t)\|_{L^\infty(D)}^\sigma \quad \text{and} \quad \|g''(u(t))\|_{L^\infty(D)} \le C \cdot \|u(t)\|_{L^\infty(D)}^{\sigma-1}$$

on $[0, T^*]$. Together with $F(u) = -g'(u)\Delta u - g''(u)|\nabla u|^2$, the above estimates finally imply

$$\begin{aligned} \|F(u(t))\|_{L^2(D)} &\le \|g'(u(t))\|_{L^\infty(D)} \cdot \|\Delta u(t)\|_{L^2(D)} \\ &\quad + \|g''(u(t))\|_{L^\infty(D)} \cdot \|\nabla u(t)\|^2_{L^4(D)} \\ &\le C \cdot \left(\varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|}\right)^\sigma \cdot e^{\sigma \cdot \lambda_\varepsilon^{\max} \cdot t} \cdot \|v_o^+\|_*^\sigma \cdot \|u(t)\|_* \\ &\quad + C \cdot \left(\varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|}\right)^\sigma \cdot e^{(\sigma+1) \cdot \lambda_\varepsilon^{\max} \cdot t} \cdot \|v_o^+\|_*^{\sigma+1} \end{aligned}$$

for all $t \in [0, T^*]$. This completes the proof. □

While (4.59) is a convenient assumption for the derivation of the nonlinearity estimate, the functional-analytic setting for (4.44) involves the Sobolev norm $\|\cdot\|_*$. The connection between the two is established by the following lemma.

LEMMA 4.5. *Assume that (A1) and (A2) are satisfied, and let $\delta^* \in (0,1)$ be arbitrary. Consider the dominating subspace X_ε^+ from (1.8) for some $\gamma_o \in (0,1)$, and let $\mathcal{G}_\varepsilon^+$ denote its subset constructed in Lemma 4.2 for $T = 1$. Then there exists a constant M such that for all sufficiently small values of $\varepsilon > 0$ the following is true.*

Let $v_o^+ \in \mathcal{G}_\varepsilon^+$ be arbitrary, and let $v^+(t) = S_\varepsilon(t) v_o^+$ denote the solution of (1.2) starting at v_o^+. Furthermore, for $0 < T^* \le 1$ assume that $u : [0, T^*] \to X^{1/2}$ is any continuous function such that for all $t \in [0, T^*]$ both

$$\|u(t)\|_* \le M \cdot \left(\varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|}\right)^{-\gamma_o} \cdot \|v_o^+\|_*^{1-\gamma_o} \tag{4.63}$$

and

$$\frac{\|u(t) - v^+(t)\|_*}{\|v^+(t)\|_*} \le \delta^* \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|} \tag{4.64}$$

hold. Then the estimate (4.61) is satisfied on $[0, T^*]$.

Proof. Using the fact that $\varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|} < 1/2$ for sufficiently small ε, the assumed inequality (4.64) and $\delta^* \in (0,1)$ imply

$$\|v^+(T^*)\|_* \le \delta^* \varepsilon^{3/2} \sqrt{|\ln \varepsilon|} \cdot \|v^+(T^*)\|_* + \|u(T^*)\|_* \le \frac{\|v^+(T^*)\|_*}{2} + \|u(T^*)\|_* ,$$

and therefore $\|v^+(T^*)\|_* \le 2 \cdot \|u(T^*)\|_*$. Combining this estimate with (4.63) and the lower bound on the growth of $\|v^+(t)\|_*$ provided by Lemma 4.1 yields

$$\|v_o^+\|_* \cdot e^{\gamma_o \cdot \lambda_\varepsilon^{\max} \cdot T^*} \le \|v^+(T^*)\|_* \le 2M \cdot \left(\varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|}\right)^{-\gamma_o} \cdot \|v_o^+\|_*^{1-\gamma_o} ,$$

as well as

$$e^{\lambda_\varepsilon^{\max} \cdot T^*} \le (2M)^{1/\gamma_o} \cdot \left(\varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|} \cdot \|v_o^+\|_*\right)^{-1} .$$

This estimate, together with (4.62) from Proposition 4.4 (which is still valid under the assumptions of this lemma), implies

$$\|u(t)\|_{L^\infty(D)} \le C \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|} \cdot \|v_o^+\|_* \cdot e^{\lambda_\varepsilon^{\max} \cdot T^*} \le C \cdot (2M)^{1/\gamma_o}$$

for all $t \in [0, T^*]$. If we now define $M = (C_3/C)^{\gamma_o}/2$, where $C_3 > 0$ is as in Proposition 4.4, then the lemma follows. □

4.3. **Spinodal Decomposition on the Disk.** In this section we study spinodal decomposition on the unit disk, i.e., we present the precise version of Theorem 1.3. For this, the following definition is essential.

DEFINITION 4.6. *Let X_ε^+ denote the dominating subspace defined in (1.8), let X_ε^- denote its orthogonal complement in the Hilbert space $X^{1/2}$, and let $\mathcal{G}_\varepsilon^+$ denote the set from Lemma 4.2 for $T = 1$. Then for every sufficiently small $\varepsilon > 0$ and for every $\delta > 0$ we define the two sets*

$$\begin{aligned} \mathcal{K}_{\varepsilon,\delta} &= \left\{ u \in X^{1/2} \ : \ \|u^-\|_* \le \delta \cdot \|u^+\|_* \right\} , \\ \mathcal{G}_{\varepsilon,\delta} &= \left\{ u \in X^{1/2} \ : \ u \in \mathcal{K}_{\varepsilon,\delta} \quad \text{and} \quad u^+ \in \mathcal{G}_\varepsilon^+ \right\} . \end{aligned}$$

where for $u \in X^{1/2}$ the elements $u^\pm$ are defined by $u = u^+ + u^- \in X_\varepsilon^+ \oplus X_\varepsilon^-$.

The cone $\mathcal{K}_{\varepsilon,\delta}$ was already recognized in [34] as a region in phase space where the nonlinearity of the Cahn-Hilliard equation remains small, even far from the equilibrium 0. The set $\mathcal{G}_{\varepsilon,\delta}$ consists of all functions in $\mathcal{K}_{\varepsilon,\delta}$ whose orthogonal projection onto the dominating subspace is contained in the set $\mathcal{G}_\varepsilon^+$. These functions constitute the collection of all initial conditions u_o for which the nonlinearity estimate can be drastically improved. We would like to point out that according to Remark 4.3, in a measure-theoretic sense the set $\mathcal{G}_\varepsilon^+$ is a large subset of the dominating subspace X_ε^+.

By combining Definition 4.6 with Lemma 4.1 and Proposition 4.4 we now obtain the main result of our paper, which is obtained similarly to [37, Theorem 4.7].

THEOREM 4.7. *Consider the Cahn-Hilliard equation (4.44). Furthermore, assume that both (A1) and (A2) are satisfied, and adopt the notation introduced in Definition 4.6. Finally, choose and fix constants $c > 0$, $\delta_o \in (0, 1/3)$ and $\varrho \in (0, 1)$.*

Then there exist ε-independent constants $d > 0$ and $\gamma_o \in (0, 1)$ (the latter determines the dominating subspace X_ε^+) such that for all sufficiently small $\varepsilon > 0$ the following holds. If $u_o \in \mathcal{G}_{\varepsilon,\delta_\varepsilon}$ with $\delta_\varepsilon = \delta_o \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|}$ is any initial condition satisfying

$$e^{-c/\varepsilon} \le \|u_o\|_* \le \min \left\{ 1 \ , \ \left(d \cdot \varepsilon^{-3/2 \cdot (1-1/\sigma)+\varrho} \right)^{1/(1-\varrho)} \right\} , \tag{4.65}$$

and if u and v denote the solutions of equations (4.44) and (1.2), respectively, starting at u_o, then there exists a first time $T_o > 0$ such that

$$\|u(T_o)\|_* = d \cdot \varepsilon^{-3/2 \cdot (1-1/\sigma)+\varrho} \cdot \|u_o\|_*^\varrho , \tag{4.66}$$

and for all $t \in [0, T_o]$ we have

$$\frac{\|u(t) - v(t)\|_*}{\|v(t)\|_*} \le \delta_\varepsilon = \delta_o \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln \varepsilon|} . \tag{4.67}$$

Proof. Choose $0 < \gamma_o < 1$ such that both

$$\frac{(\sigma+1)(1-\gamma_o)}{\sigma+1-\gamma_o} < \varrho \quad \text{and} \quad \frac{-3\gamma_o \cdot (\sigma-1)}{2(\sigma+1-\gamma_o)} < -\frac{3}{2} \cdot \left(1 - \frac{1}{\sigma} \right) + \varrho \tag{4.68}$$

hold. Furthermore, choose $d_1 > 0$ such that for all sufficiently small $\varepsilon > 0$ we have

$$d_1 \cdot \varepsilon^{-3/2 \cdot (1-1/\sigma)+\varrho} \le \left(\varepsilon^{-3/2} \cdot |\ln \varepsilon|^{-1/2} \right)^{\gamma_o \cdot (\sigma-1)/(\sigma+1-\gamma_o)} . \tag{4.69}$$

The constant d appearing in the formulation of the theorem is defined as $d = d_1 \cdot d_2$, where the precise choice of $d_2 > 0$ will be made later in the proof.

Let $\gamma_\varepsilon = \gamma_o \cdot \lambda_\varepsilon^{\max}$, and adopt the notation of Lemma 4.1. Choose an initial condition u_o as in the formulation of the theorem and define $v_o^+ = u_o^+$. Then the orthogonal projection of the solution $v(t)$ onto X_ε^+ is given by $v^+(t) = S_\varepsilon(t)v_o^+$. Furthermore, the choice of u_o implies $v_o^+ \in \mathcal{G}_\varepsilon^+$, and with $\delta^* = 3\delta_o < 1$ we obtain

$$\frac{\|u(0) - v^+(0)\|_*}{\|v^+(0)\|_*} = \frac{\|u_o - v_o^+\|_*}{\|v_o^+\|_*} = \frac{\|u_o^-\|_*}{\|u_o^+\|_*} \le \frac{\delta^*}{3} \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln\varepsilon|} \; .$$

Invoking the continuity of u and v^+, we can now let $T^* \in (0,1]$ be the maximal time such that for all $t \in [0,T^*]$ we have both

$$\frac{\|u(t) - v^+(t)\|_*}{\|v^+(t)\|_*} \le \delta^* \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln\varepsilon|} \tag{4.70}$$

and

$$\|u(t)\|_* \le d \cdot \varepsilon^{-3/2\cdot(1-1/\sigma)+\varrho} \cdot \|u_o\|_*^\varrho \; , \tag{4.71}$$

where $d = d_1 \cdot d_2$. Notice also that according to (4.65), the right-hand side of (4.71) is strictly larger than $\|u_o\|_*$. For $d_2 \le M/2$, where M denotes the constant from Lemma 4.5, the definition of d and (4.69) furnish for sufficiently small ε

$$d \cdot \varepsilon^{-3/2\cdot(1-1/\sigma)+\varrho} \le \frac{M}{2} \cdot \left(\varepsilon^{-3/2} \cdot |\ln\varepsilon|^{-1/2}\right)^{\gamma_o} \; ,$$

and due to the choice of u_o and $\delta_o < 1/3$ we further obtain

$$\|u_o\|_* = \sqrt{\|u_o^+\|_*^2 + \|u_o^-\|_*^2} \le \sqrt{1 + \delta_o^2 \cdot \varepsilon^3 \cdot |\ln\varepsilon|} \cdot \|u_o^+\|_* \le 2 \cdot \|u_o^+\|_* \; . \tag{4.72}$$

Together with $u_o^+ = v_o^+$, (4.65), and (4.68) this implies

$$\|u_o\|_*^\varrho \le 2 \cdot \|v_o^+\|_*^\varrho \le 2 \cdot \|v_o^+\|_*^{(\sigma+1)\cdot(1-\gamma_o)/(\sigma+1-\gamma_o)} \le 2 \cdot \|v_o^+\|_*^{1-\gamma_o} \; .$$

Consequently, the validity of (4.71) on $[0,T^*]$ implies (4.63). Due to Lemma 4.5 all the assumptions of Proposition 4.4 are therefore satisfied on $[0,T^*]$. Due to the definition of u and v, the variation of constants formula gives

$$u(t) - v(t) = \int_0^t S_\varepsilon(t-s)F(u(s))\,ds \; ,$$

and Lemma 4.1 and Proposition 4.4 furnish

$$\begin{aligned}
\|u(t) - v(t)\|_* &\le \int_0^t K_\varepsilon \cdot (t-s)^{-1/2} \cdot e^{\beta_\varepsilon\cdot(t-s)} \cdot \|F(u(s))\|_{L^2(D)}\,ds \\
&\le K_\varepsilon \cdot C_4 \cdot \varepsilon^{3\sigma/2} \cdot |\ln\varepsilon|^{\sigma/2} \cdot \|v_o^+\|_*^{\sigma+1} \cdot e^{(\sigma+1)\cdot\lambda_\varepsilon^{\max}\cdot t} \\
&\qquad \cdot \int_0^t (t-s)^{-1/2} \cdot e^{(\beta_\varepsilon - (\sigma+1)\cdot\lambda_\varepsilon^{\max})\cdot(t-s)}\,ds \\
&\le K_\varepsilon \cdot C_4 \cdot \varepsilon^{3\sigma/2} \cdot |\ln\varepsilon|^{\sigma/2} \cdot \|v_o^+\|_*^{\sigma+1} \cdot e^{(\sigma+1)\cdot\lambda_\varepsilon^{\max}\cdot t} \\
&\qquad \cdot \left((\sigma-\beta_o)\cdot\lambda_\varepsilon^{\max}\right)^{-1/2} \cdot \int_0^\infty s^{-1/2} \cdot e^{-s}\,ds
\end{aligned}$$

for all $t \in [0,T^*]$, as long as $\beta_o < 1$. Thus, due to Lemma 4.1 and (1.6) there exists an ε-independent constant $C_o > 0$ such that

$$\|u(t) - v(t)\|_* \le C_o \cdot \varepsilon^{3\sigma/2} \cdot |\ln\varepsilon|^{\sigma/2} \cdot \|v_o^+\|_*^{\sigma+1} \cdot e^{(\sigma+1)\cdot\lambda_\varepsilon^{\max}\cdot t}$$

for all $t \in [0,T^*]$. According to Lemma 4.1 we have $\|v^+(t)\|_* \ge e^{\gamma_\varepsilon\cdot t} \cdot \|v_o^+\|_*$, and with $\gamma_\varepsilon = \gamma_o \cdot \lambda_\varepsilon^{\max}$ one obtains

$$\frac{\|u(t) - v(t)\|_*}{\|v^+(t)\|_*} \le C_o \cdot \varepsilon^{3\sigma/2} \cdot |\ln\varepsilon|^{\sigma/2} \cdot \|v_o^+\|_*^\sigma \cdot e^{(\sigma+1-\gamma_o)\cdot\lambda_\varepsilon^{\max}\cdot t} \; . \tag{4.73}$$

Since $u_o - v_o^+ = u_o^- \in X_\varepsilon^-$ we have $\|S_\varepsilon(t)(u_o - v_o^+)\|_* \le e^{\gamma_\varepsilon \cdot t} \cdot \|u_o - v_o^+\|_*$, and Lemma 4.1 yields

$$\frac{\|v(t) - v^+(t)\|_*}{\|v^+(t)\|_*} = \frac{\|S_\varepsilon(t)(u_o - v_o^+)\|_*}{\|S_\varepsilon(t)v_o^+\|_*} \le \frac{\|u_o - v_o^+\|_*}{\|v_o^+\|_*} \le \frac{\delta^*}{3} \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln\varepsilon|} \quad (4.74)$$

for all $t \in [0, T^*]$.

Next we need to establish an upper bound on T^*, which in turn allows us to bound the relative distance in (4.73). For small ε, (4.70) and $\delta^* \in (0,1)$ imply

$$\|v^+(T^*)\|_* \le \delta^* \cdot \varepsilon^{3/2}\sqrt{|\ln\varepsilon|} \cdot \|v^+(T^*)\|_* + \|u(T^*)\|_* \le \frac{\|v^+(T^*)\|_*}{2} + \|u(T^*)\|_* \,,$$

i.e., we have $\|v^+(T^*)\|_* \le 2 \cdot \|u(T^*)\|_*$. Together with (4.68), (4.69), (4.71), (4.72), and Lemma 4.1, this yields

$$\begin{aligned} e^{\lambda_\varepsilon^{\max} \cdot T^*} &\le \left(\frac{\|v^+(T^*)\|_*}{\|v_o^+\|_*}\right)^{1/\gamma_o} \le \left(4d_2 d_1 \varepsilon^{-3/2\cdot(1-1/\sigma)+\varrho} \cdot \|v_o^+\|_*^{\varrho-1}\right)^{1/\gamma_o} \\ &\le (4d_2)^{1/\gamma_o} \cdot \left(\varepsilon^{3/2}|\ln\varepsilon|^{1/2}\right)^{(1-\sigma)/(\sigma+1-\gamma_o)} \cdot \|v_o^+\|_*^{-\sigma/(\sigma+1-\gamma_o)}, \quad (4.75) \end{aligned}$$

which is the desired upper bound for T^*. With (4.73) this implies

$$\frac{\|u(t) - v(t)\|_*}{\|v^+(t)\|_*} \le C_o \cdot (4d_2)^{(\sigma+1-\gamma_o)/\gamma_o} \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln\varepsilon|} \quad \text{for all} \quad t \in [0, T^*] \,.$$

Now we choose the constant d_2. According to the discussion following (4.71) we certainly need $d_2 \le M/2$. Assume additionally that $C_o \cdot (4d_2)^{(\sigma+1-\gamma_o)/\gamma_o} \le \delta^*/3$ is satisfied. Then the above estimate and (4.74) furnish

$$\frac{\|u(t) - v(t)\|_*}{\|v(t)\|_*} \le \frac{\|u(t) - v(t)\|_*}{\|v^+(t)\|_*} \le \frac{\delta^*}{3} \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln\varepsilon|} = \delta_o \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln\varepsilon|} \,,$$

as well as

$$\frac{\|u(t) - v^+(t)\|_*}{\|v^+(t)\|_*} \le \frac{2}{3} \cdot \delta^* \cdot \varepsilon^{3/2} \cdot \sqrt{|\ln\varepsilon|}$$

for all $t \in [0, T^*]$. This shows that (4.67) holds everywhere on $[0, T^*]$. Moreover, since $T^* \in (0, 1]$ is the maximal time for which (4.70) and (4.71) hold on $[0, T^*]$, we have to have $\|u(T^*)\|_* = d \cdot \varepsilon^{-3/2\cdot(1-1/\sigma)+\varrho} \cdot \|u_o\|_*^\varrho$, unless $T^* = 1$. Yet, (4.75) and the assumed lower bound on $\|u_o\|_*$ imply $T^* = O(\varepsilon)$ for $\varepsilon \to 0$, i.e., $T^* < 1$ as long as $\varepsilon > 0$ is sufficiently small — and the theorem follows with $T_o = T^*$. ☐

Remark 4.8. It was pointed out in [37, Remark 4.8] that the assumed lower bound on $\|u_o\|_*$ could easily be relaxed. However, initial conditions which are exponentially close to the homogeneous equilibrium are not physically reasonable; see also the discussion in [28].

For the case of total mass $\mu = 0$ and the standard cubic nonlinearity $f(u) = u - u^3$ the above theorem guarantees linear behavior in the Cahn-Hilliard equation (1.1) on the disk up to distances of the order $R_\varepsilon \sim \varepsilon^{-3/4+\varrho}$. In fact, up to this point the relative distance between the solution to the nonlinear and the linearized equation is of the order $O(\varepsilon^{3/2} \cdot |\ln\varepsilon|^{1/2})$. But does the result really describe the behavior exhibited by generic random initial conditions? In the case of rectangular domains, the analogue of Theorem 4.7 really does reproduce the typical solution behavior, as was shown numerically in [37]. Yet, for the unit disk the situation is more complicated.

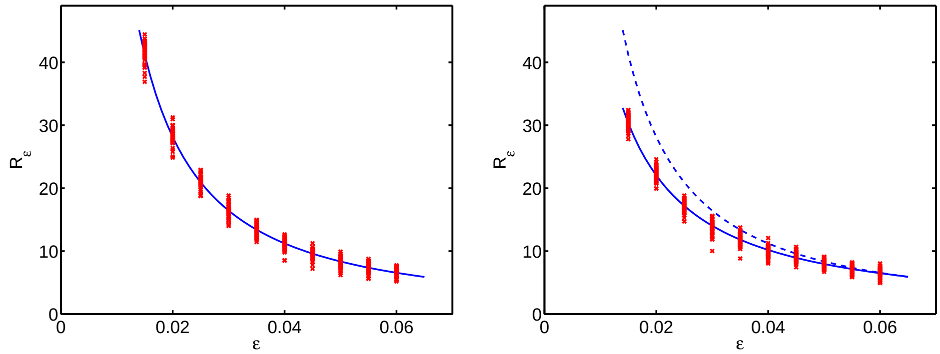

FIGURE 3. Dependence of the radius R_ε on ε for random initial conditions starting near $\bar{u}_o = 0$ on the disk. The left diagram is for relative distance $0.2 \cdot \varepsilon^{1.5} \cdot \sqrt{|\ln \varepsilon|}$; a least-squares fit gives the dependence $R_\varepsilon \approx 0.16 \cdot \varepsilon^{-1.3}$. The right diagram is for relative distance $0.78 \cdot \varepsilon^{1.8}$; a least-squares fit now gives $R_\varepsilon \approx 0.28 \cdot \varepsilon^{-1.1}$.

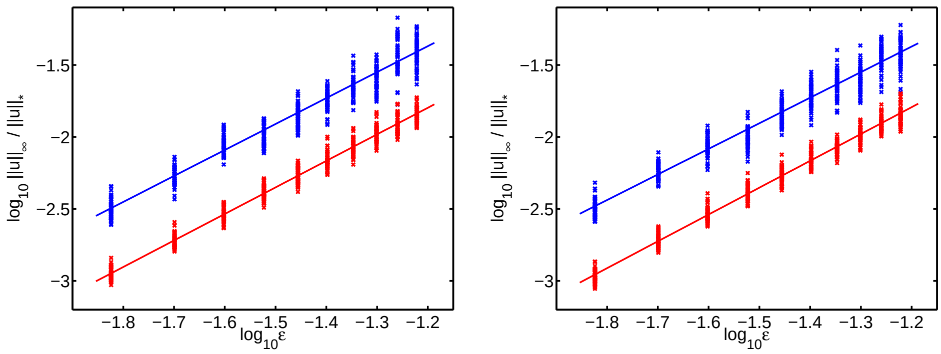

FIGURE 4. Dependence of the quotient $\delta_\varepsilon = \|u\|_\infty / \|u\|_*$ on ε for the random initial conditions used in Figure 3, as well as for the function u at the end of the simulation. The left diagram is for relative distance $0.2 \cdot \varepsilon^{1.5} \cdot \sqrt{|\ln \varepsilon|}$; a least-squares fit gives the dependence $\delta_\varepsilon \approx 6.2 \cdot \varepsilon^{1.80}$ for the initial conditions (top curve), and $\delta_\varepsilon \approx 2.6 \cdot \varepsilon^{1.84}$ at the simulation end (bottom curve). The right diagram is for relative distance $0.78 \cdot \varepsilon^{1.8}$; a least-squares fit gives the dependence $\delta_\varepsilon \approx 5.8 \cdot \varepsilon^{1.78}$ for the initial conditions (top curve), and $\delta_\varepsilon \approx 2.8 \cdot \varepsilon^{1.86}$ at the simulation end (bottom curve).

The left diagram in Figure 3 shows the result of simulations for various ε-values, starting at fifty randomly chosen initial conditions in each case. For each initial condition, we followed the corresponding evolution of the nonlinear and the linearized Cahn-Hilliard equation until the relative distance between these solutions reached $\delta_\varepsilon = 0.2 \cdot \varepsilon^{1.5} \cdot \sqrt{|\ln \varepsilon|}$, and recorded the resulting $\|\cdot\|_*$-norm of the nonlinear solution as R_ε. A least-squares fit of these data points reveals that $R_\varepsilon \approx 0.16 \cdot \varepsilon^{-1.3}$, which is considerably larger than the order predicted by Theorem 4.7. In fact, repeating these simulations with $\delta_\varepsilon = 0.78 \cdot \varepsilon^{1.8}$ leads to $R_\varepsilon \approx 0.28 \cdot \varepsilon^{-1.1}$, which is still larger than the predicted order — albeit closer.

These simulations clearly indicate that, unlike in the situation of rectangular domains, the nonlinearity estimate derived in Proposition 4.4 is suboptimal. Since one of the main ingredients of this nonlinearity estimate is a tight bound on the ratio $\|u\|_{L^\infty(D)}/\|u\|_*$, we determined these ratios numerically for both the initial conditions and the solutions at the end of the simulations in Figure 3. The results are shown in the log-log-plots in Figure 4 — and they give rise to two interesting conclusions:

- The bound on $\|u\|_{L^\infty(D)}/\|u\|_*$ derived in (4.53) is suboptimal. In fact, it seems that $\|u\|_{L^\infty(D)}/\|u\|_* \sim \varepsilon^{1.8}$ describes the observed behavior better.
- The norm ratios actually decrease as the solution evolves, which most likely results in even better nonlinearity estimates.

If we assume that the ratio mentioned in the first conclusion is more accurate, then an adaptation of Theorem 4.7 would predict $R_\varepsilon \sim \varepsilon^{-0.9}$ — in contrast to the order $R_\varepsilon \sim \varepsilon^{-1.1}$ that was derived numerically in the right diagram of Figure 3. However, the difference between the exponents is much smaller now, and can probably be explained by the second conclusion from above.

This discussion shows that in the case of the unit disk, random superpositions alone cannot completely describe the linear solution behavior observed during spinodal decomposition, even though the remaining gap is small. We conjecture that this difference is a consequence of the eigenfunction representations given in Lemma 3.1. The maxima of the eigenfunctions are attained at different points in the domain, which generically leads to smaller $L^\infty(D)$-norms of eigenfunction superpositions than predicted by the results of Section 2. In contrast, for rectangular domains all eigenfunctions achieve their maximum at the origin, i.e., additional cancellations are less likely.

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