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Fixed points indices and period-doubling cascades

Madhura R. Joglekar, Evelyn Sander and James A. Yorke

Dedicated to Steve Smale

Abstract. Period-doubling cascades are among the most prominent features of many smooth one-parameter families of maps, $F : \mathbb{R} \times \mathfrak{M} \to \mathfrak{M}$, where $\mathfrak{M}$ is a locally compact manifold without boundary, typically $\mathbb{R}^N$. In particular, we investigate $F(\mu, \cdot)$ for $\mu \in J = [\mu_1, \mu_2]$, when $F(\mu_1, \cdot)$ has only finitely many periodic orbits while $F(\mu_2, \cdot)$ has exponential growth of the number of periodic orbits as a function of the period. For generic F, under additional hypotheses, we use a fixed point index argument to show that there are infinitely many "regular" periodic orbits at μ_2. Furthermore, all but finitely many of these regular orbits at μ_2 are tethered to their own period-doubling cascade. Specifically, each orbit ρ at μ_2 lies in a connected component $C(\rho)$ of regular orbits in $J \times \mathfrak{M}$; different regular orbits typically are contained in different components, and each component contains a period-doubling cascade. These components are one-manifolds of orbits, meaning that we can reasonably say that an orbit ρ is "tethered" or "tied" to a unique cascade. When $F(\mu_2)$ has horseshoe dynamics, we show how to count the number of regular orbits of each period, and hence the number of cascades in $J \times \mathfrak{M}$.

As corollaries of our main results, we give several examples, we prove that the map in each example has infinitely many cascades, and we count the cascades.

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1. Introduction

In Figure 1, as μ increases towards a value $\mu_F \approx 3.57$, the logistic map has a family of periodic orbits that undergoes an infinite sequence of period doublings with the period of these orbits tending to ∞. Such a family with periods going to infinity is called a cascade. A period-doubling cascade was first reported by Myrberg in 1962 [25]. The existence of many such cascades is one

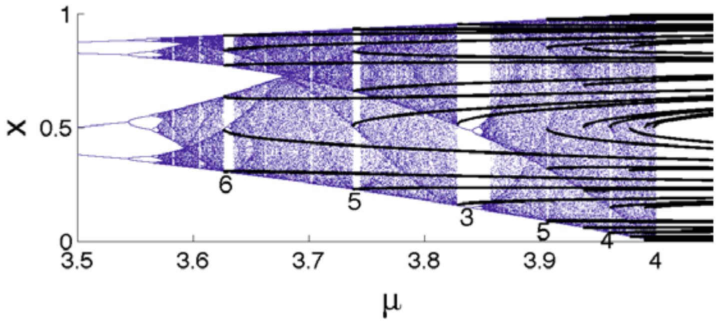

Figure 1. Cascades and their connected components. An attracting set for the logistic map $F(\mu, x) = \mu x(1-x)$ is shown in blue. There are infinitely many cascades, each with infinitely many period-doubling bifurcations. Each saddle-node bifurcation creates both a cascade and a path of regular unstable orbits (shown in black for up to period 6).

of the most prominent features observed in the study of parametrized maps. Robert May popularized their existence to a huge scientific audience [23]. They are found numerically and experimentally in a large number of scientific contexts and are often associated with the onset of chaos. For examples, see [1, 2, 3, 4, 10, 13, 14, 15, 16, 17, 22, 31, 32, 33, 35].

The quadratic map $\mu - x^2$ has an infinite number of cascades, which follows from the monotonicity result of Milnor and Thurston [24], which states that for this map, periodic orbits are never destroyed as μ increases. Orbits are created at saddle-node bifurcations and the attractor branch created there can only undergo supercritical period-doubling bifurcations. This family of maps has no inverted saddle-node bifurcations and no inverted period-doubling bifurcations. Without special behavior such as monotonicity, showing that there are infinitely many cascades is much more difficult.

Once a cascade is known to exist, it can be understood using the elegant scaling and renormalization theory found in the work from the early 1980s of Feigenbaum [12] and others [5, 6, 7, 8, 9, 11, 21] as well as some more recent results in [19, 20] (see also the references in [20]). This theory in particular focuses on the "Feigenbaum" scaling of a typical cascade. For example there is a so-called Feigenbaum number $\delta \sim 4.66920$ so that if μ_n is the parameter value of the nth period-doubling bifurcation of a cascade, and $\mu_\infty = \lim_{n\to\infty} \mu_n$, then for a typical cascade,

$$|\mu_n - \mu_\infty| \sim k\delta^{-n} \quad \text{for some } k > 0. \tag{1}$$

These works have shown that the value of δ given above is a universal constant which does not in general depend on the family. This is described in more detail in Section 6.

Considering the beauty and successes of the several approaches to cascades from the 1980s, one might wonder the purpose of reconsideration of the question of cascades thirty years later. The reason is thus, there is a gap in the classical theory. Specifically, this theory does not explain when cascades exist in the first place. In this paper, we answer this question and in addition we provide a means for enumerating cascades as a function of the periodic orbit from which they arise. The classical methods of renormalization theory are not well suited for understanding existence and enumeration. Instead, we use methods of topological index theory.

From a scientific point of view, one might imagine that an existence proof for cascades is an unnecessary mathematical argument, since so often the existence of cascades is clear from a simple numerical simulation. But our theory shows that many higher-dimensional cascades are unstable and thus could not be found easily using numerical methods. Specifically, under quite weak assumptions, we show that cascades—potentially unstable ones—occur whenever there is an onset of chaos. Thus the understanding of cascades is critical to the understanding of chaos. Furthermore, our ability to enumerate cascades gives a new tool for classifying complexity.

There are only a few results about the existence of cascades [34] for general systems. In several recent papers [29, 28, 27, 30], we have described the results of a new general theory of cascades, which explains why cascades exist and why chaotic dynamical systems often have infinitely many cascades. However, each of these results has relied on being able to count the number of *regular* (a.k.a. *nonflip*) periodic orbits. We show here that it suffices to count only the number of periodic orbits. To this end, the main result of Section 2 is the Regular Periodic Orbits Theorem (Theorem 3), which says that the number of regular periodic orbits is approximately half of the total number of periodic orbits. In particular, if there are infinitely many of one, there are infinitely many of the other.

The paper proceeds as follows. Section 2 begins with definitions and prior results, ending with the Regular Periodic Orbits Theorem (Theorem 3). In Section 3, we concentrate on the question of quantification. We start by giving results on counting the number of period-k points in a generalized high-dimensional horseshoe and end by using the abstract results to count the number of period-k cascades for each integer k.

In Section 4 we examine the scalar map

$$S_\mu(x) = \mu \sin(2\pi x) \bmod 1.$$

This map has the interesting property that its topological entropy is unbounded as $\mu \to \infty$. We show that for each period p, the number of cascades having period-p orbits is unbounded. The key lemma we prove for this map is that for positive integer values of μ, the map $S_\mu(x)$ is conjugate to an expanding map $T_\mu(y)$, that is, $\inf |dT/dy| > 1$.

In Section 5, we give an example of a map in which chaos forms without cascades and a conjecture that this is not typical. We end in Section 6 by giving further background of the history of cascades research.

2. Finding cascades

In this section, we give a series of definitions involved in our study of cascades. We describe our prior results on cascades, using the number of regular periodic orbits. The first main result of this paper is to show that we can drop assumptions on the number of regular periodic orbits. Specifically, we show that the number of regular periodic is approximately half of the number of periodic orbits.

2.1. Preliminaries

We investigate smooth (i.e., C^∞) maps $F : J \times \mathfrak{M} \to \mathfrak{M}$ where J is an interval and $\mathfrak{M}$ is any smooth manifold $\mathfrak{M}$ of finite dimension. We write $F(\mu, x)$ where $\mu \in J$ and $x \in \mathfrak{M}$.

We say that a point (μ, x_0) is a *period-p point* if $F^p(\mu, x_0) = x_0$ and p is the smallest positive integer for which that is true. Its *orbit*, sometimes written $[(\mu, x_0)]$, is the set

$$\{(\mu, x_0), (\mu, x_1), \ldots, (\mu, x_{p-1})\}, \quad \text{where } x_j = F^j(\mu, x_0).$$

By the *eigenvalues* of a period-p point (μ, x_0) or of its orbit, we mean the eigenvalues of the Jacobian matrix $D_x F^p(\mu, x_0)$.

An orbit is called *hyperbolic* if none of its eigenvalues has absolute value 1. All other orbits are *bifurcation orbits*.

We call a periodic orbit a *flip* orbit if the orbit has an odd number of eigenvalues less than -1, and -1 is not an eigenvalue. (When $\mathfrak{M}$ is one dimensional, this condition is $\partial F^p/\partial x(\mu, x) < -1$. In dimension two, flip orbits are those with exactly one eigenvalue less than -1.) All other periodic orbits are called *regular*. For typical F, a period-doubling orbit is one that has an eigenvalue equal to -1. As μ increases, a path of periodic orbits of constant period that passes through a typical period-doubling bifurcation orbit will switch between flip and regular orbits at that bifurcation orbit since an eigenvalue crosses -1.

There is an equivalent way to define a period-p orbit $[(\mu, x_0)]$ to be a "flip" orbit. Let E be the maximum unstable subspace of the tangent space $T_x(\mu, x_0)$. That is, E is the tangent plane of the unstable manifold (in $\mathfrak{M}$) of (μ, x_0). Then (μ, x_0) is flip exactly when it is hyperbolic and the $D_x F^p(\mu, x_0)$ is orientation reversing on E; that is, the determinant of $D_x F^p$ on E is negative.

For some interval K, assume $Y : K \to J \times \mathfrak{M}$ is continuous. Write $Y(\psi) = (\mu(\psi), x(\psi))$ where $\psi \in K$. We say that Y is a (regular) *path* if

(i) $Y(\psi)$ is a regular periodic point for each $\psi \in K$;
(ii) Y does not retrace orbits; that is, Y is never in the same orbit for different ψ.

Let $I \subset J$ and $M \subset \mathfrak{M}$, where I is an interval. We write $\mathrm{RPO}(I \times M)$ for the set of regular periodic orbits with $(\mu, x) \in I \times M$. We say that a path $Y(\cdot)$ is *maximal in $I \times M$* if the following additional condition holds:

(iii) Y cannot be extended further to a larger interval; that is, it cannot be redefined to include more points of $\mathrm{RPO}(I \times M)$.

Note that $[Y(\cdot)]$ is in effect a path in RPO, and it is continuous when we put the Hausdorff metric on RPO.

We call a (regular) path $Y(\cdot) \subset \mathrm{RPO}(I \times M)$ a *cascade in* $I \times M$ if the domain of Y is a half-open interval, say $[a, b)$, and the path contains infinitely many period-doubling bifurcations, and for some period p, the set of periods of the points in the path is precisely the unbounded set $\{p, 2p, 4p, 8p, \ldots\}$. As one traverses the cascade, the periods need not increase monotonically, but $\liminf$ of the period of $Y(\psi)$ is ∞ as $\psi \to b$.

Write $\mathrm{fixed}(\mu, p)$ for the set of fixed points of $F^p(\mu, \cdot)$, and $\mathrm{Per}(\mu, p)$ for the set of period-p points of F. Write $|\,\mathrm{fixed}(\mu, p)|$ and $|\,\mathrm{Per}(\mu, p)|$, respectively, for the *number* of those points. Correspondingly, let $\mathrm{rpf}(\mu, p)$ denote the set of regular periodic points which are fixed points of F^p, hence, $(\mathrm{rpf}(\mu, p) \subset \mathrm{fixed}(\mu, p))$, and let $\rho(\mu, p)$ denote the set of regular period-p points $(\rho(\mu, p) \subset \mathrm{Per}(\mu, p))$.

We say that there is an *exponential periodic orbit growth* at μ if there is a number $G > 1$ for which $|\,\mathrm{fixed}(\mu, p)| \geq G^p$ for infinitely many p. For example, this inequality might hold for all even p, but for odd p there might be no periodic orbits. This is equivalent to

$$h = \limsup_{p \to \infty} \frac{\log |\,\mathrm{fixed}(\mu, p)|}{p} \geq \log G > 0. \tag{2}$$

Call h in the above equation the *periodic orbit entropy.* If $h < \infty$, the sequence (p_j) is an *entropy achieving sequence* if $\lim_{j \to \infty} p_j = \infty$ and this $\limsup$ is achieved by the sequence. That is,

$$\lim_{j \to \infty} \frac{\log |\,\mathrm{fixed}(\mu, p_j)|}{p_j} = h. \tag{3}$$

Entropy achieving sequences are guaranteed to exist.

We say that a map $F(\mu, \cdot)$ has *periodic orbit (PO) chaos* at a parameter μ if there is exponential periodic orbit growth. This occurs whenever there is a horseshoe for some iterate of the map. It is sufficiently general to include having one or multiple coexisting chaotic attractors, as well as the case of nonattracting chaotic orbits. As hinted at by equation (2), in many cases PO chaos is equivalent to positive topological entropy.

The *unstable dimension* $\mathrm{Dim}_u(\mu, x_0)$ of a periodic point (μ, x_0) or periodic orbit is defined to be the number of its eigenvalues λ having $|\lambda| > 1$, counting multiplicities. (We always count multiplicities of an eigenvalue λ by considering the Jacobian $D_x F^p(\mu, x_0)$ in Jordan canonical form and counting the number of occurrences of λ on the main diagonal. For Dim_u we count the number of diagonal entries with absolute value greater than 1.)

We say there is *virtually uniform PO chaos* at μ if there is PO chaos, and all but a finite number of periodic orbits have the same unstable dimension, denoted by $\mathrm{Dim}_u(\mu)$; x_0 is omitted to indicate that it is independent of x_0.

Generic maps. Our results are given for generic maps of a parameter. Specifically, we say that the map F is *generic* if all of the bifurcation orbits are *generic*, meaning that each bifurcation orbit is one of the following three types.

1. A standard saddle-node bifurcation. (Where "standard" means the form of the bifurcation stated in a standard textbook, such as Robinson [26].) In particular, the orbit has only one eigenvalue λ for which $|\lambda| = 1$, namely $\lambda = 1$.
2. A standard period-doubling bifurcation. In particular, the orbit has only one eigenvalue λ for which $|\lambda| = 1$, namely $\lambda = -1$.
3. A standard Hopf bifurcation. In particular, the orbit has only one complex pair of eigenvalues λ for which $|\lambda| = 1$. We require that these eigenvalues are not roots of unity; that is, there is no integer $k > 0$ for which $\lambda^k = 1$.

Each regular hyperbolic orbit O is locally contained in a unique path of periodic orbits. Furthermore, O has a neighborhood N in the Hausdorff metric topology in which all orbits in N are on that path. In particular, the connected component C of the orbit in RPO is locally a one-manifold near 0.

Our motivation for considering generic bifurcations is that this same property also holds, if O is, instead, a regular bifurcation orbit. That is, let C denote the connected component of the orbit O in RPO. Then O has a neighborhood N for which $C \cap N$ is a one-manifold.

Each generic regular bifurcation periodic orbit has a neighborhood N in which all RPOs in N are locally contained in a unique path of RPOs. Hence the connection to cascades can be summarized as follows: starting at each regular periodic orbit Q for $\mu \in [\mu_1, \mu_2]$, there is a local path of regular periodic orbits through Q. Enlarge this path as far as possible, either the path reaches μ_1 or μ_2, or there is a cascade.

2.2. Theorems about infinitely many cascades

We start by listing our assumptions, and then state our main theorems.

List of assumptions.

(A_0) Assume F is a generic smooth map; that is, F is infinitely differentiable in μ and x, and all of its bifurcation orbits are generic.

(A_1) Let $I = [\mu_1, \mu_2]$, where $\mu_1 < \mu_2$. Assume there is a bounded set M that contains all periodic points (μ, x) for $\mu \in I$.

(A_2) Assume all periodic orbits at μ_1 and μ_2 are hyperbolic.

(A_3) Assume that the number Λ_1 of periodic orbits at μ_1 is finite.

(A_4) Assume that at μ_2, there is virtually uniform PO chaos. Write Λ_2 for the number of periodic orbits at μ_2 having unstable dimension not equal to $\mathrm{Dim}_u(\mu_2)$.

Theorem 1. *Assume* (A_0)–(A_4). *Then there are infinitely many distinct period-doubling cascades in* $I \times \mathfrak{M}$.

Before proceeding, we reformulate Theorem 1 in a less compact but more comprehensible way, making it clearer from whence the cascades appear.

Theorem 2. *Assume* (A_0)–(A_4). *Define* $S = I \times \mathfrak{M}$. *Then the following are true.*

(B1) *There are infinitely many regular periodic points at* μ_2.

(B2) *For each maximal path* $Y(\psi) = (\mu(\psi), x(\psi))$ *in* S *starting from a regular periodic point* $Y_0 = (\mu_2, x_0)$, *the set of traversed orbits, denoted by* Orbits(Y), *depends only on the initial orbit containing* Y_0. *That is, different initial points on the same orbit yield paths that traverse the same set of orbits, so we can write* Orbits(Y_0) *for* Orbits(Y).

(B3) *Let* $Y_0 = (\mu_2, x_0)$ *and* $Y_1 = (\mu_2, x_1)$ *be regular periodic points on different orbits. Then* Orbits(Y_0) *and* Orbits(Y_1) *are disjoint.*

(B4) *Let* K *denote the unstable dimension of a regular periodic point* (μ_2, x_0). *For a maximal path* $Y(\psi) = (\mu(\psi), x(\psi))$ *in* S *starting from* (μ_2, x_0), *let* $k(\psi)$ *denote the unstable dimension of* $Y(\psi)$. *At each direction-reversing bifurcation,* $k(\psi)$ *changes parity; that is, it changes from odd to even or vice versa. Initially* $Y(a) = (\mu_2, x_0)$, *so initially* $\mu(\psi)$ *is decreasing and* $k(a) = K$, *so initially* $k(a) + K$ *is even. Hence, in general,* $\mu(\psi)$ *is decreasing if* $K + k(\psi)$ *is even and increasing if it is odd.*

(B5) *Let* Y *be a maximal path on* $[a, b]$ *in* S *and* $\mu(a) = \mu_2$. *If* $\mu(b) = \mu_2$, *then* $\mu(\psi)$ *is increasing at* $\psi = b$, *but decreasing at* $\psi = a$. *Hence* $k(a) + k(b)$ *is odd, so* $k(a) \neq k(b)$.

(B6) *There are infinitely many distinct period-doubling cascades on* $[\mu_1, \mu_2] \times \mathfrak{M}$.

(B7) *There are at most* $\Lambda = \Lambda_1 + \Lambda_2$ *regular periodic orbits at* μ_2, *with unstable dimension* $\mathrm{Dim}_u(\mu_2)$, *that are not connected to cascades.*

This theorem was proved in [29], under the additional assumption that at μ_2, there are infinitely many regular periodic points. The main result of this section shows that the additional assumption holds automatically under assumptions (A_0)–(A_4).

Definition 1 (Asymptotic regularity). We say that *asymptotically half of the periodic points are regular* at μ if for every entropy achieving sequence (p_j),

$$\lim_{j\to\infty} \frac{|\,\mathrm{rpf}(\mu, p_j)|}{|\,\mathrm{fixed}(\mu, p_j)|} = \frac{1}{2}. \tag{4}$$

We now state the main theorem of this section.

Theorem 3 (Regular Periodic Orbits). *Under assumptions* (A_0)–(A_4), *asymptotically half of the periodic points are regular at* μ_2. *In particular, at* μ_2, *there are infinitely many regular periodic orbits.*

The next subsection contains a proof of this theorem, after introducing the required topological index theory.

2.3. Fixed point index and proof of the main theorem

We now introduce the fixed point index of a hyperbolic periodic point. Assume $F^p(\mu, x) = x$ and the orbit of x is hyperbolic. In particular, the period of (μ, x) is either p or a divisor of p. Let M be the Jacobian matrix $D_x F^p(\mu, x)$. Define $\mathrm{Pos}(\mu, x, p)$ to be the number of eigenvalues (counting multiplicities) of M that are greater than $+1$. Define $\mathrm{Index}_p(\mu, x) = (-1)^{\mathrm{Pos}(\mu,x,p)}$. Let (μ, x) be a period p_0 point. Then

$$\mathrm{Index}_{kp_0}(\mu, x) = -\mathrm{Index}_{p_0}(\mu, x) \tag{5}$$

if and only if k is even and the orbit of (μ, x) is flip.

If (μ, x) is a periodic point, write $\mathrm{Per}(\mu, x)$ for its (minimum) period. We will use the shorthand $\mathrm{Index}_{\mathrm{Per}}(\mu, x)$ for $\mathrm{Index}_k(\mu, x)$ where $k = \mathrm{Per}(\mu, x)$.

For any finite subset S of $\mathrm{fixed}(\mu, p)$, define the fixed point index of S to be $\mathrm{Index}(S) = \Sigma_{x\in S} \mathrm{Index}_p(\mu, x)$. When $|\,\mathrm{fixed}(\mu, p)| < \infty$, we use the shorthand notation $\mathrm{Index}(\mu, p)$ for $\mathrm{Index}(\mathrm{fixed}(\mu, p))$. Define *average fixed point index* of S to be

$$\langle \mathrm{Index}(S)\rangle = \frac{\Sigma_{x\in S} \mathrm{Index}_p(\mu, x)}{|S|}. \tag{6}$$

With the notation developed, we now prove Theorem 3.

Proof of Theorem 3. Assume (A_0)–(A_4), and let (p_j) be an entropy achieving sequence for F at μ_2. We consider two cases: the case in which all p_j are odd, and the case in which some p_j are even.

Odd p_j case. Assume first that it is possible to choose all periods p_j to be odd.

Assumptions (A_1) and (A_2) imply that $\mathrm{fixed}(\mu, p)$ is finite for each p. In particular, $\mathrm{Index}(\mu_1, p)$ and $\mathrm{Index}(\mu_2, p)$ are well defined. Assumptions (A_0) and (A_1) imply that $\mathrm{Index}(\mu_1, p) = \mathrm{Index}(\mu_2, p)$ for each period p. Since the total number of periodic orbits at μ_1 is finite, there is some integer

$$B_1 = \max_{1\le p<\infty} |\,\mathrm{Index}(\mu_1, p)|.$$

That B_1 is finite is a straightforward fact since we assume that the periodic orbits at μ_1 are hyperbolic and since the index of each periodic orbit is ± 1 for each p. Homotopy invariance [18] implies that for each period p,

$$\mathrm{Index}(\mu_1, p) = \mathrm{Index}(\mu_2, p),$$

and in particular,

$$|\,\mathrm{Index}(\mu_2, p)| \le B_1 \quad \text{for some } B_1.$$

For the sequence (p_j), the number of fixed points goes to infinity. Thus the number of points with index $+1$ is nearly equal to those with index -1, the difference being at most B_1. Hence

$$\langle \mathrm{Index}(\mu_2, p_j)\rangle \to 0.$$

Notice that since p_j is odd, for each fixed point x of $F^{p_j}(\mu_2, \cdot)$, we have

$$\text{Index}_{\text{Per}}(\mu_2, x) = \text{Index}_{p_j}(\mu_2, x).$$

Since the average tends to 0, the fraction of fixed points of $F^{p_j}(\mu_2, \cdot)$ that has index $+1$ tends to half, as $j \to \infty$, as does the fraction with index -1. If two orbits have the same unstable dimension Dim_u and one has index $+1$ and the other has index -1, then one has an odd number of eigenvalues less than -1 and the other has an even number. Hence one is contained in a regular orbit and the other is contained in a flip orbit. Since by (A_4) all but a finite number of orbits at μ_2 have the same unstable dimension, we see that asymptotically half the periodic points are regular, and that as $|\,\text{fixed}(\mu, p_j)| \to \infty$, there must be infinitely many regular periodic orbits.

Even p_j case. Assume now that it is not possible to choose all the periods (p_j) to be odd. Then there are only finitely many that are odd. After discarding all odd p_j from the sequence, we can assume all p_j are even. In this case, some of the points in $\text{fixed}(\mu, p)$ may have period k, for which p/k is even. Hence k also divides $p/2$. That means that such a point is also a fixed point in $\text{fixed}(\mu, p/2)$. Only for such points is it possible that,

$$\text{Index}_{\text{Per}}(\mu_2, x) \neq \text{Index}_{p_j}(\mu_2, x).$$

Since the growth rate G_0 has been chosen to be as large as possible, we have

$$\frac{|\,\text{fixed}(\mu, \frac{p_j}{2})|}{|\,\text{fixed}(\mu, p_j)|} \to 0 \quad \text{as } j \to \infty.$$

Hence as $j \to \infty$, such points have negligible effect on the calculations in the odd case, and we can again conclude that asymptotically half of the periodic points are regular, and that there are infinitely many regular orbits at μ_2. □

3. Counting cascades

The previous section included results on the existence of cascades. This section develops methods to count them. We start in the first subsection by giving an abstract set of results on counting the number of regular periodic orbits for an m-shift. Since the number of regular periodic orbits is in one-to-one correspondence with the number of cascades, we are able to apply these results to counting cascades. We do so in the following subsection.

3.1. Counting regular orbits of an m-shift

In this section, we give a recursive formula for the number of regular periodic orbits for an m-shift. A formula for the two-shift or tent map appears in [28] using a bifurcation theory argument, but the general case has not previously been shown.

Definition 2 (Regular periodic orbits for the m-shift). Consider the full shift on m symbols. That is, let Σ_m be the space of all possible infinite sequences of m symbols, and let σ_m be the shift map on Σ_m. In addition to the standard sequence, we additionally define a mapping Sgn with range ± 1 on each of the

m fixed points of Σ_m. If the value of Sgn on the fixed point $(s, s, s, \ldots)$ is positive, we refer to s as a *positive symbol*. Likewise, if $\mathrm{Sgn}((s, s, s, \ldots)) = -1$, we refer to s as a *negative symbol*. The terms positive and negative are intended to call to mind that the derivative at the point is positive (or in higher dimensions orientation preserving) or negative (orientation reversing in higher dimensions). If there are j positive symbols and $m - j$ negative symbols, we denote the shift mapping by $\sigma_{(m,j)}$, as shorthand for the existence of the mapping Sgn. We now extend Sgn to periodic orbits: let $(s_1, s_2, \ldots)$ be a periodic point under $\sigma_{(m,j)}$. That is, for some (not necessarily least) k, $s_{j+k} = s_j$ for all $j \geq 1$. Let r be the number of symbols in the set $(s_1, \ldots, s_k)$ (with multiplicity) which are negative. We define $\mathrm{Sgn}(s, k) = (-1)^r$.

Let $a = (a_1, a_2, \ldots)$ be a (least) period-k point of $\sigma_{(m,j)}$. The point a is called a *regular periodic point* if $\mathrm{Sgn}(s, k) = 1$. A periodic point which is not regular is called a *flip periodic point*. These definitions are the symbol analogue of the definitions given for a map in a hyperbolic set on a smooth manifold.

Definition 3 (Counting notation). We define a series of quantities, some of which we will use only later. (The subscript s denotes that these are the symbol analogue of a quantity already defined for a smooth map.)

$$\begin{aligned}
\mathrm{Per}_s(m, j, k) &= \text{the set of period-}k\text{ points of } \sigma_{(m,j)};\\
\rho_s(m, j, k) &= \text{the set of regular period-}k\text{ points of } \sigma_{(m,j)};\\
\mathrm{Fix}_s(m, j, k) &= \text{the set of fixed points of } \sigma^k_{(m,j)};\\
e(m, j, k) &= \text{the set of fixed points of } \sigma^k_{(m,j)} \text{ such that } \mathrm{Sgn}(\cdot, k) = 1;\\
\theta(m, j, k) &= \text{the set of fixed points of } \sigma^k_{(m,j)} \text{ such that } \mathrm{Sgn}(\cdot, k) = -1.
\end{aligned}$$

Note that $\mathrm{Per}_s(m, j, k)$ and $\mathrm{Fix}_s(m, j, k)$ are independent of j and that

$$\mathrm{Per}_s(m, j, k) = \rho_s(m, m, k) \quad \text{and} \quad \mathrm{Fix}_s(m, j, k) = e(m, m, k).$$

Furthermore, $\mathrm{Per}_s(m, j, k) \subset \mathrm{Fix}_s(m, j, k)$. In addition, since each fixed point s of $\sigma^k_{(m,j)}$ has either $\mathrm{Sgn}(s, k) = 1$ or $\mathrm{Sgn}(s, k) = -1$, then

$$\mathrm{Fix}_s(m, j, k) = e(m, j, k) \cup \theta(m, j, k).$$

By standard counting arguments, $|\mathrm{Fix}_s(m, j, k)| = m^k$. Thus

$$|\theta(m, j, k)| = m^k - |e(m, j, k)|.$$

Theorem 4 (Recursive counting formula for RPOs of an m-shift). *Let*

$$\mathrm{Odd}(k) = \{n < k : k/n \text{ is odd}\}.$$

If k is odd, then

$$|\rho_s(m, j, k)| = |e(m, j, k)| - \sum_{n \in \mathrm{Odd}(k)} |\rho_s(m, j, n)|,$$

and if k is even, then

$$|\rho_s(m,j,k)| = |e(m,j,k)| - \sum_{n\in \mathrm{Odd}(k)} |\rho_s(m,j,n)| - m^{k/2},$$

where $|\cdot|$ denotes the number of elements in the set.

Proof. Define

$$\mathrm{Even}(k) = \{n < k : k/n \text{ is even}\}.$$

Note that $\mathrm{Even}(k)$ is empty if k is odd. We proceed by showing the following.

(a) The following formula holds:

$$|\rho_s(m,j,k)| = |e(m,j,k)| - \sum_{n\in \mathrm{Odd}(k)} |\rho_s(m,j,n)| - \sum_{n\in \mathrm{Even}(k)} |\operatorname{Per}_s(m,j,n)|.$$

(b) If k is even, then $\sum_{n\in \mathrm{Even}(k)} |\operatorname{Per}_s(m,j,n)| = |\operatorname{Fix}_s(m,j,k/2)|$.

The results of the theorem follow from (a) and (b), along with the fact noted above, that for all k, $|\operatorname{Fix}_s(m,j,k)| = m^k$.

Proof of (a). We know that $\rho_s(m,j,k) \subset e(m,j,k)$. The elements of

$$e(m,j,k) \setminus \rho_s(m,j,k)$$

are exactly those elements which are fixed under $\sigma^k_{(m,j)}$, with the least period less than k. Every point fixed under $\sigma^k_{(m,j)}$ is of the form $(\overline{a_1,\ldots,a_n})$, a period-$n$ point (where the overbar denotes repetition to form a forward infinite sequence), where $k/n = r$, since the concatenation of r copies of the length-n sequence $(a_1,\ldots,a_n)$ is a length-k sequence. If r is even, then concatenating r copies of any period-n sequence yields a length-k sequence with an even number of negative symbols. That is, it is an element of $e(m,j,k)\setminus\rho_s(m,j,k)$. However, if r is odd, then r copies of a length-n sequence a yields a length-k sequence with an odd number of negative symbols, indicating that the fixed point of $\sigma^k_{(m,j)}$ is contained in $e(m,j,k)\setminus\rho_s(m,j,k)$, only when a is a regular periodic orbit.

Proof of (b). Assume that k is even. Let X be a period-n orbit, where $n \in \mathrm{Even}(k)$, and define p by the formula $k/n = 2p$. Then X is in an element of $\operatorname{Fix}_s(m,j,k/2)$. Furthermore, as shown in the proof of part (a), every element of $\operatorname{Fix}_s(m,j,k/2)$ is either a period-$k/2$ orbit, or is a period-n orbit, where n divides $k/2$. But n divides $k/2$ exactly when $k/(2n) = p$, which is the condition for being a member of $\mathrm{Even}(k)$. Therefore, there is a one-to-one correspondence between the elements of $\operatorname{Per}_s(m,j,n)$ for $n \in \mathrm{Even}(k)$ and the elements of $\operatorname{Fix}_s(m,j,k/2)$. □

The only quantity left to compute is $e(m,j,k)$. The following lemma gives an algorithmic formula for computing $e(m,j,k)$.

Lemma 1 (Computing $e(m,j,k)$). *For the case $k = 1$,*

$$|e(m,j,1)| = j \quad \text{and} \quad |\theta(m,j,1)| = m - j.$$

For $k > 1$,

$$|e(m,j,k)| = j \cdot |e(m,j,k-1)| + (m-j) \cdot |\theta(m,j,k-1)|$$

and

$$|\theta(m,j,k)| = j \cdot |\theta(m,j,k-1)| + (m-j) \cdot |e(m,j,k-1)|.$$

Proof. The $k = 1$ case is clear, since there are j positive symbols and $m - j$ negative symbols. We can get all elements of $e(m,j,k)$ by adding a symbol to the right of a length-$(k-1)$ sequence of symbols Ψ in such a way that there are an even number of negative symbols in the resulting sequence of k symbols. That is, one of the following occurs:

(e1) $\Psi \in e(m,j,k-1)$ and the added symbol is positive, or
(e2) $\Psi \in \theta(m,j,k-1)$ and the added symbol is negative.

Likewise, $\theta(m,j,k)$ is formed by adding one symbol to a length-$(k-1)$ sequence Ψ in such a way that one of the following occurs:

(θ1) $\Psi \in \theta(m,j,k-1)$ and the added symbol is positive, or
(θ2) $\Psi \in e(m,j,k-1)$ and the added symbol is negative. □

Recall that for any k, $|\theta(m,j,k)| = m^k - |e(m,j,k)|$. We combine both facts into a formula only containing e as follows:

$$|e(m,j,k)| = j \cdot |e(m,j,k-1)| + (m-j) \cdot (m^{k-1} - |e(m,j,k-1)|).$$

The next lemma gives a general relationship between $e(m,k,j)$ and $\theta(m,k,j)$.

Lemma 2. *Assume that*

$$j = (m+\ell)/2.$$

This is equivalent to $j - (m-j) = \ell$, meaning that ℓ is the difference between the number of positive symbols and the number of negative symbols. The quantity ℓ can either be positive or negative. Then

$$|e(m,j,k)| = (m^k + \ell^k)/2,$$

or equivalently,

$$|e(m,j,k)| - |\theta(m,j,k)| = \ell^k.$$

Proof. The equivalence of the two different forms of the formula is straightforward. We prove the second one. The formula holds for $k = 1$, since

$$|e(m,j,1)| - |\theta(m,j,1)| = j - (m-j) = \ell.$$

Proceeding inductively, assume $|e(m,j,n)| - |\theta(m,j,n)| = \ell^n$ for all $n < k$.

$$\begin{aligned} |e(m,j,k)| - |\theta(m,j,k)| &= (j \cdot |e(m,j,k-1)| + (m-j) \cdot |\theta(m,j,k-1)|) \\ &\quad - (j \cdot |\theta(m,j,k-1)| + (m-j) \cdot |e(m,j,k-1)|) \\ &= (j-(m-j)) \cdot \ell^{k-1} = \ell^k. \end{aligned}$$ □

This algorithm can be automated using a symbolic computation package. We include such a code in the Appendix. The resulting $|\rho_s|$ values for a number of m, j, and k are tabulated in Table 1.

TABLE 1. The number of regular period-k orbits for an m-shift, with j positive symbols for all values of $2 \le m \le 5$, $0 \le j \le m$, and $k \le 10$.

Values of $|\rho_s(m, j, k)|$

(m, j)	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
(2,0)	0	1	0	3	0	9	0	30	0	99
(2,1)	1	0	1	1	3	4	9	14	28	48
(2,2)	2	1	2	3	6	9	18	30	56	99
(3,0)	0	3	0	18	0	116	0	810	0	5880
(3,1)	1	1	4	8	24	56	156	400	1092	2928
(3,2)	2	1	4	8	24	56	156	400	1092	2928
(3,3)	3	3	8	18	48	116	312	810	2184	5880
(4,0)	0	6	0	60	0	670	0	8160	0	104754
(4,1)	1	3	9	30	99	335	1161	4080	14532	52377
(4,2)	2	2	10	28	102	330	1170	4064	14560	52326
(4,3)	3	3	11	30	105	335	1179	4080	14588	52377
(4,4)	4	6	20	60	204	670	2340	8160	29120	104754
(5,0)	0	10	0	150	0	2580	0	48750	0	976248
(5,1)	1	6	16	82	288	1340	5424	24746	107408	490920
(5,2)	2	4	20	72	312	1280	5580	24336	108500	487968
(5,3)	3	4	20	72	312	1280	5580	24336	108500	487968
(5,4)	4	6	24	82	336	1340	5736	24746	109592	490920
(5,5)	5	10	40	150	624	2580	11160	48750	217000	976248

The case of m-tent maps. We now consider our counting in the special case in which j is as close to $m/2$ as possible. This occurs for example for one-dimensional m-tent maps, in which every other crossing of the origin has a negative derivative. The map can start with either a positive or a negative derivative close to zero. For m even, this distinction does not change the number of crossings for which the derivative is negative, but for m odd, it does. However, in the next lemma we show that this distinction only changes the number of regular periodic points for $k = 1$.

Lemma 3. *If m is odd, then for every $k > 1$,*

$$\left|\rho_s\left(m, \frac{m-1}{2}, k\right)\right| = \left|\rho_s\left(m, \frac{m+1}{2}, k\right)\right|.$$

Proof. Let $j_- = (m-1)/2$ and $j_+ = (m+1)/2$. Clearly

$$|\rho_s(m, j_\pm, 1)| = j_\pm$$

and

$$|\rho_s(m, j_\pm, 2)| = \frac{m^2 + (\pm 1)^2}{2} - 2^1 = \frac{m^2+1}{2} - 2.$$

Inductively, assume equality for all $n < k$. Consider the expression for $|\rho_s(m, j_\pm, k)|$ given in Theorem 4. By our inductive hypothesis, there are only two terms in that expression that could possibly differ. They are $|e(m, j_\pm, k)|$ and $|\rho_s(m, j_\pm, 1)|$. There are two cases.

If k is even, then 1 is not contained in $\mathrm{Odd}(k)$, and

$$|e(m, j_\pm, k)| = \frac{m^k + (\pm 1)^k}{2},$$

which is the same in both cases. This completes the proof of the even k case.

If k is odd, then 1 is contained in $\mathrm{Odd}(k)$. Further,

$$|e(m, j_\pm, k)| - |\rho_s(m, j_\pm, 1)| = \frac{m^k + (\pm 1)^k}{2} - \frac{m \pm 1}{2} = \frac{m^k - m}{2}.$$

Therefore, $|\rho_s|$ is the same for j_+ and j_-. This completes the proof of the odd k case. □

The techniques of this proof lead to the following statement for general m and j (not restricted to tent maps).

Lemma 4. *For all m, if k is even, then for any $\ell < m$ such that $(m-\ell)/2$ is an integer,*

$$\left|\rho_s\left(m, \frac{m-\ell}{2}, k\right)\right| = \left|\rho_s\left(m, \frac{m+\ell}{2}, k\right)\right|.$$

Proof. Define $j_\pm = (m \pm \ell)/2$. For $k = 2$, $|\rho_s(m, j_\pm, 2)| = (m^2 + (\pm\ell)^2)/2 - m$ and are thus equal for j_+ and j_-. For any even k,

$$|e(m, j_\pm, k)| = \frac{m^k + (\pm\ell)^k}{2},$$

and are thus equal for j_+ and j_-. If k is even, then every $n \in \mathrm{Odd}(k)$ is also even. Therefore the result follows by induction. □

3.2. Counting cascades for smooth maps

We now use the results of the previous subsection to *count* the infinitely many cascades shown to exist in Theorems 1 and 2. There are guaranteed to be infinitely many cascades under the hypotheses of these theorems, but they fall into distinct categories which are enumerable. Assume that F satisfies assumptions (A_0)–(A_4). There are two fundamentally different types of cascades on $[\mu_1, \mu_2] \times M$: cascades which are unique in their maximal paths on $[\mu_1, \mu_2] \times M$, called *solitary cascades*, and cascades which are not unique in their maximal path in $[\mu_1, \mu_2] \times M$, in which case their maximal path contains exactly two cascades. Such cascades are called *paired cascades* (cf. [30]).

The maximal path of a paired cascade is contained entirely within the interior of $[\mu_1, \mu_2] \times M$ and never intersects the boundary. We have shown previously that if $\mu_0 < \mu_1 < \mu_2$ are such that (A_0)–(A_4) are satisfied on the parameter intervals $[\mu_0, \mu_1]$ and $[\mu_2, \mu_1]$, a condition we call *off-on-off chaos*, then *all but a finite number of cascades are paired.* Paired cascades are sensitive to perturbations and are easy to create and destroy. Therefore counting them is of little interest, since they are not dynamically stable in any reasonable sense.

In contrast to the paired case, solitary cascades are robust to large changes in the map. Thus these are the cascades which we count. Solitary cascades are most familiar in the dynamical literature, as all cascades of quadratic maps are solitary. Theorems 1 and 2 only describe solitary cascades, showing that the maximal path of a solitary cascade contains exactly one regular orbit on the boundary of the parameter region (i.e., at $\mu = \mu_1$ or at $\mu = \mu_2$). We refer to the period of this regular orbit as the stem period of the cascade on $[\mu_1, \mu_2]$. In the examples in the current section, we can find a value μ_2^* sufficiently large such that for any $\kappa > \mu_2$, each cascade has the same stem period on $[\mu_1, \mu_2^*]$ and on $[\mu_1, \kappa]$. In the next section, we will discuss a map such that cascades continue to form as $\mu \to \infty$.

Definition 4 (Dynamical horseshoe). Assume that F satisfies assumptions (A_0)–(A_4). Assume that at μ_2, there is a hyperbolic invariant set of F which is topologically conjugate to the m-shift with j positive symbols and $m - j$ negative symbols. Assume that the conjugacy sends points in any orbit with an orientation-reversing Jacobian to negative symbols and points in an orbit with orientation-preserving Jacobian to positive symbols. We call this set a dynamical $\{m, j\}$ horseshoe for $F(\mu_2, \cdot)$.

The following lemma links the previous section to the number of cascades.

Lemma 5. *Assume that F satisfies assumptions (A_0)–(A_4), that there are no periodic orbits at μ_1, and that at μ_2, there is a hyperbolic invariant set for F, which is a dynamical $\{m, j\}$ horseshoe, and there are no periodic orbits outside the hyperbolic invariant set. Then the number of stem period-k solitary cascades is $|\rho_s(m, j, k)|$, as computed in the previous section.*

Proof. This is a corollary of Theorem 2. Note that $\Gamma_1 = 0$ by assumption. Since all periodic orbits are contained in a hyperbolic invariant set of $F(\mu_2, \cdot)$, they all have the same unstable dimension, and $\Gamma_2 = 0$. Thus every solitary cascade corresponds to a unique regular point at μ_2. The topological conjugacy maps regular period-k points of F to regular period-k points of $\sigma_{(m,j)}$ (and flip period-k points of F to flip period-k points of $\sigma_{(m,j)}$). Therefore, the number of regular period-k points for F at μ_2 is equal to $|\rho_s(m, j, k)|$ as computed in the previous section. □

Counting stem period-k cascades. Lemma 5 is quite general, in that it applies to any horseshoe. We use it to compute cascades in a few specific examples.

One-dimensional examples. We now present three illustrative examples.

$$Quad(\mu, x) = \mu - x^2 + g(\mu, x), \tag{7}$$

$$Cubic(\mu, x) = \mu x - x^3 + g(\mu, x), \tag{8}$$

$$Quart(\mu, x) = x^4 - 2\mu x^2 + \mu^2/2 + g(\mu, x), \tag{9}$$

where for some real positive β,

$$\begin{aligned} |g(\mu,0)| &< \beta \quad \text{for all } \mu, \\ |g_x(\mu,x)| &< \beta \quad \text{for all } \mu, x. \end{aligned} \tag{10}$$

For μ sufficiently negative, (7) and (9) have no periodic orbits. For μ sufficiently negative, (8) has only one periodic point and it is a flip fixed point. Hence (8) has no regular orbits.

For large μ, all three equations have dynamical horseshoes and all their periodic points are part of their horseshoe. See [29] for the detailed calculations for the first two maps, and [30] for the third. The horseshoes are, respectively, $\{2,1\}$, $\{3,1\}$, and $\{4,2\}$ horseshoes. Thus for a residual set of g, the number of stem period-k solitary cascades for each can be read directly off the corresponding rows of Table 1 in the previous section. Although the existence of infinitely many cascades has been shown in [29], the current paper is the first to compute these values.

Two-dimensional examples. Similarly, we have shown in [28] that large perturbations of the Hénon map are planar systems such that for small parameters, there are no periodic orbits, and for large values of μ, there is a $\{2,1\}$ dynamical horseshoe. Specifically, we refer to the system

$$H(\mu,x_1,x_2) = \begin{pmatrix} \mu + \beta x_2 - x_1^2 + g(\mu,x_1,x_2) \\ x_1 + h(\mu,x_1,x_2) \end{pmatrix}, \tag{Hénon}$$

where β is a nonzero fixed value, and the added function (g,h) is smooth and is very small for $\|(\mu,x_1,x_2)\|$ sufficiently large. Thus, for a residual set of (g,h), the number of stem period-k solitary cascades is equal to $|\rho_s(2,1)|$ in Table 1.

N-dimensional examples. Another example is an N-dimensional coupled system of equations, namely,

$$x_i \mapsto K_i(\mu) - x_i^2 + g_i(x_1,\ldots,x_N), \tag{Coupled Quadratic}$$

where $g = (g_1,\ldots,g_N)$ is bounded with bounded first derivatives, and

$$\lim_{k\to\pm\infty} K_i(\mu) = \pm\infty.$$

In such a system, if μ is small, there are no periodic orbits. If μ is large, there is a $\{2^N, 2^{N-1}\}$ dynamical horseshoe. Thus for a residual set of g, the number of stem period-k solitary cascades is given by $|\rho_s(2^N,2^{N-1})|$. For example, for the planar case, the number of cascades is equal to the number of cascades for the quartic class of map *Quart* in the one-dimensional section above.

4. An example with ever-increasing topological entropy

In our previous examples, there is a finite parameter interval containing all bifurcations. In this section we give an example such that the number of

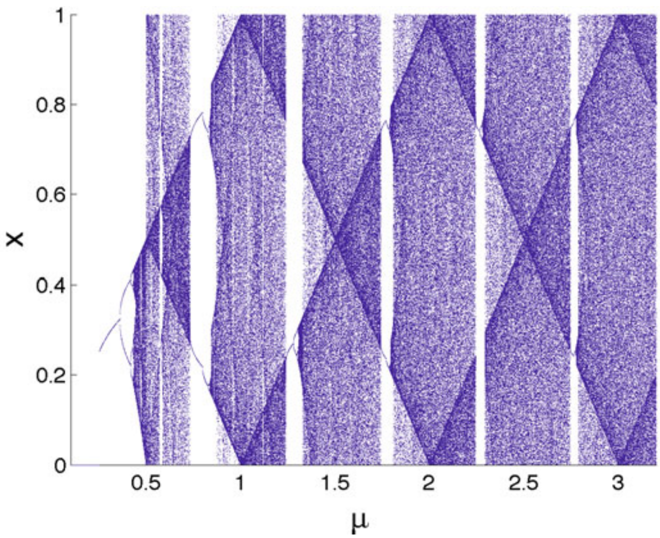

FIGURE 2. A bifurcation diagram for S_μ, showing a portion of the attracting set for each μ. Cascades continue to form as μ goes to infinity. Note that for $\mu \in [0, \frac{1}{2\pi}), x = 0$ is stable. For some μ, there are multiple attractors. The union of the attractors is symmetric about $x = \frac{1}{2}$ for each μ. The apparent asymmetry of the figure reveals that another attractor exists but is not shown. See, for example, the cascade near $\mu = 0.4$; this is paired with another cascade which is not shown.

cascades continues to increase for arbitrarily large values of the parameter. Specifically, we investigate the map $S_\mu : [0,1] \to [0,1)$ defined by

$$S_\mu(x) := \mu \sin(2\pi x) \bmod 1.$$

The bifurcation diagram is shown in Figure 2. The map is depicted in Figures 3 and 4.

As the parameter μ is increased, many dynamical systems reach a level of maximum topological entropy followed by a decrease to zero entropy. Consider the forced damped pendulum and the forced Duffing equation, where T is the period and μ is the strength of the forcing. The time T map for both the differential equations has that property; there is no chaos for large $|\mu|$. For the quadratic map and the Hénon map, the entropy reaches a maximum, namely ln 2, and is thereafter, a constant. For this system, however, the entropy increases without bound as $\mu \to \infty$. In fact, the number of cascades in the system and consequently the chaos in the system increases as μ increases, and the system has an ever-increasing number of solitary cascades for $\mu \in [0, m]$ as m is increased to higher and higher integer values. Specifically, we calculate the number of period-k solitary cascades, for $k > 1$,

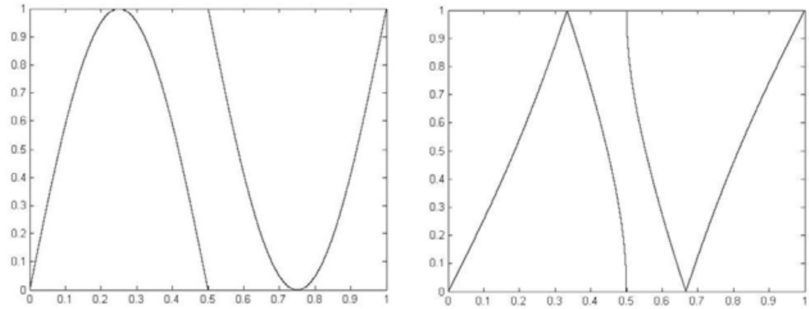

FIGURE 3. $S_\mu(x)$ and its conjugate map $T_\mu(y)$ for $\mu = 1$.

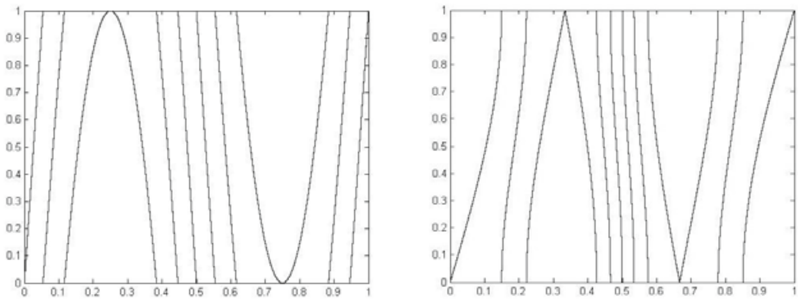

FIGURE 4. $S_\mu(x)$ and its conjugate map $T_\mu(y)$ for $\mu = 3$.

for positive integer values of $\mu \in [0, m]$, where $m \in \mathbb{N}$. As in the previous examples, each solitary cascade in $\mu \in [0, m]$ is contained in a component of $\mathrm{RPO}([0, m] \times [0, 1])$ that includes a regular orbit at $\mu = m$. We do not rule out the possibility that a solitary cascade on $[0, m]$ would be a paired cascade on $[0, \infty)$. In that case, the cascade it is paired with would have μ value greater than m. In order to complete these calculations, we show that for positive integer values of μ, S_μ is conjugate to a map T_μ for which $\inf |T'_\mu| > 1$. Thus $S_\mu(x)$ is unstable for $\mu > 0$ an integer. This allows us to conclude that for each positive integer μ, the map S_μ is chaotic, and every periodic orbit is unstable. We can apply Lemma 5 to enumerate the number of cascades in the system at a given value of μ.

The map. For each positive integer μ, i.e., $\mu \in \mathbb{N}$, the map is rotationally symmetric with respect to point $(\frac{1}{2}, \frac{1}{2})$. The interval $[0, 1)$ is divided into 4μ parts, each of which maps onto $[0, 1)$.

The map $C : [-1, 1] \to [0, 1]$ given by

$$C(y) = \frac{1 - \cos(\pi y)}{2}$$

has been used to establish a conjugacy between the logistic map and the tent map.

We define the map $T_\mu := C^{-1}(S_\mu(C))$ where

$$C^{-1}(x) = \frac{\cos^{-1}(1-2x)}{\pi}.$$

Lemma 6. *For $\mu \in \mathbb{N}$, the conjugate map T_μ satisfies* $\inf |T'_\mu| > 1$.[1]

Proof.

$$\begin{aligned} T_\mu(y) &= C^{-1}(S_\mu(C(y))) \\ &= \frac{\cos^{-1}(1 - 2\,[\,\mu \sin(\pi\cos(\pi y)) \bmod 1])}{\pi}. \end{aligned} \tag{11}$$

Hence,

$$\begin{aligned} \frac{dT_\mu(y)}{dy} &= \frac{2(\mu\sin(\pi\cos(\pi y)) \bmod 1)'}{\pi\sqrt{(1-(1-2[\mu\sin(\pi\cos(\pi y)) \bmod 1])^2}} \\ &= \frac{(\mu\sin(\pi\cos(\pi y)) \bmod 1)'}{\pi\sqrt{(\mu\sin(\pi\cos(\pi y)) \bmod 1) - (\mu\sin(\pi\cos(\pi y)) \bmod 1)^2}} \\ &= \frac{(\mu\sin(\pi\cos(\pi y)))'}{\pi\sqrt{(\mu\sin(\pi\cos(\pi y)) \bmod 1) - (\mu\sin(\pi\cos(\pi y)) \bmod 1)^2}}, \end{aligned} \tag{12}$$

where $'$ denotes derivative. Therefore,

$$\left|\frac{dT_\mu(y)}{dy}\right|^2 = \frac{[(\mu\sin[\pi\cos(\pi y)])']^2}{\pi^2[(\mu\sin(\pi\cos(\pi y)) \bmod 1) - (\mu\sin(\pi\cos(\pi y)) \bmod 1)^2]}$$

since the "mod 1" does not affect the derivative of the function at a point, so long as the derivative is defined at that point.

The derivative $dT_\mu(y)/dy$ is defined everywhere on $[0,1]$ except for a finite number of points. In this section, our notation will ignore the fact that we are taking infima of functions which are not defined at a finite number of points.

Define $L_\mu = \inf_{y\in(0,\frac{1}{2})} |dT_\mu(y)/dy|$. Considering $y \in [0,1]$, ${T_\mu}'(y) = {T_\mu}'(1-y)$. Hence, ${T_\mu}'(y)$ is symmetric with respect to the line $y = \frac{1}{2}$. Hence, it is enough to prove that $L_\mu > 1$ which is equivalent to showing that ${L_\mu}^2 > 1$. For convenience, write $\theta(y) = \theta = \pi\cos(\pi y)$ for $y \in (0, \frac{1}{2})$ which gives $\theta \in (0,\pi)$. Rewrite $[\mu\sin(\pi\cos(\pi y)) \bmod 1] = [\mu\sin\theta \bmod 1]$ as $(\mu\sin\theta - k)$ where k is an integer such that $0 \le (\mu\sin\theta - k) < 1$, i.e., k is the integer part of $\mu\sin\theta$. Depending on $\theta \in (0,\pi)$, k ranges from 0 to $\mu - 1$.

Thus, we prove

$${L_\mu}^2 = \inf_{\theta\in(0,\pi)} \frac{\mu^2(\pi^2-\theta^2)\cos^2\theta}{(\mu\sin\theta - k) - (\mu\sin\theta - k)^2} > 1. \tag{13}$$

[1]Note that $|\frac{dT_\mu(0)}{dy}| = \sqrt{2\pi\mu}$ which appears to be $\inf_{y\in(0,\frac{1}{2})} |\frac{dT_\mu(y)}{dy}|$

Case 1: $\mu = 1$**.** For $\mu = 1$, $k = 0$.

$$\begin{aligned} {L_1}^2 &= \inf_{\theta\in(0,\pi)} \frac{(\pi^2-\theta^2)\cos^2\theta}{\sin\theta - \sin^2\theta} \\ &= \inf_{\theta\in(0,\pi)} \frac{(1+\sin\theta)(1-\sin\theta)(\pi^2-\theta^2)}{\sin\theta(1-\sin\theta)} \\ &= \inf_{\theta\in(0,\pi)} \left(1+\frac{1}{\sin\theta}\right)(\pi^2-\theta^2). \end{aligned} \tag{14}$$

However,

$$\inf_{\theta\in(0,\pi)} \left(1+\frac{1}{\sin\theta}\right)(\pi^2-\theta^2) > \inf_{\theta\in(0,\pi)} \left(\frac{1}{\sin\theta}\right)(\pi-\theta).$$

Hence, it is enough to show that

$$\left(\frac{1}{\sin\theta}\right)(\pi-\theta) \geq 1 \quad \text{for } \theta \in (0,\pi),$$

which is equivalent to showing that

$$g(\theta) = \pi - \theta - \sin\theta \geq 0 \quad \text{for } \theta \in (0,\pi). \tag{15}$$

Now, $g'(\theta) = -1 - \cos\theta < 0$ for $\theta \in (0,\pi)$. Hence, $g(\theta)$ is a decreasing function for $\theta \in (0,\pi)$, so g attains a minimum as $\theta \to \pi^-$ and $\lim_{\theta\to\pi^-} g(\theta) = 0$. Therefore, (15) holds, and hence

$${L_1}^2 = \inf_{\theta\in(0,\pi)} \frac{(\pi^2-\theta^2)\cos^2\theta}{\sin\theta - \sin^2\theta} > 1. \tag{16}$$

This finishes the case $\mu = 1$.

Figure 3 shows the graph of $S_\mu(x)$ and the corresponding graph of $T_\mu(y)$ for $\mu = 1$.

Case 2: $\mu > 1$**.** Assume $\mu \in \mathbb{N}$, $\mu > 1$. In this case, we show that

$${L_\mu}^2 = \inf_{\theta\in(0,\pi)} \frac{\mu^2(\pi^2-\theta^2)\cos^2\theta}{(\mu\sin\theta - k) - (\mu\sin\theta - k)^2} > 1.$$

From (16), it is enough to show that for $\theta \in (0,\pi)$,

$$\frac{\mu^2(\pi^2-\theta^2)\cos^2\theta}{(\mu\sin\theta - k) - (\mu\sin\theta - k)^2} \geq \frac{(\pi^2-\theta^2)\cos^2\theta}{\sin\theta - \sin^2\theta} \tag{17}$$

$$\begin{aligned} &\iff \frac{\mu^2}{(\mu\sin\theta - k) - (\mu\sin\theta - k)^2} \geq \frac{1}{\sin\theta - \sin^2\theta} \\ &\iff \mu^2(\sin\theta - \sin^2\theta) \geq (\mu\sin\theta - k) - (\mu\sin\theta - k)^2 \\ &\iff \mu^2\sin\theta \geq \mu\sin\theta - k - k^2 + 2\mu k\sin\theta \\ &\iff (\mu^2 - \mu - 2\mu k)\sin\theta \geq -k - k^2. \end{aligned} \tag{18}$$

Note that k is an integer and a function of θ, that is, $k = k(\theta)$ such that $0 \leq \mu\sin\theta - k < 1$. Since $\sin\theta$ is a continuous function, $k(\theta)$ is piecewise continuous with jump discontinuities. Also,

$$(\mu^2 - \mu - 2\mu k)\sin\theta \geq (\mu^2 - \mu - 2\mu k).$$

Hence, from (18), it is enough to show that $(\mu^2 - \mu - 2\mu k) \geq -k - k^2$ which is equivalent to showing that

$$(\mu - k)^2 \geq (\mu - k),$$

which is always true since $(\mu - k)$ is an integer. This proves (17). Hence, $\inf_{y\in(0,\frac{1}{2})} |T'_\mu(y)| > 1$ for $\mu > 1$, when μ is an integer. □

Figure 4 shows the graphs of $S_\mu(x)$ and the corresponding graphs of $T_\mu(y)$ for $\mu = 3$.

The above lemma shows that every periodic orbit of the map $T_\mu(y)$ is unstable, and since $T_\mu(y)$ and $S_\mu(x)$ are conjugates, every periodic orbit of the map $S_\mu(x) = \mu \sin(2\pi x) \bmod 1$ is unstable, for positive integer values of μ.

Counting cascades for S_μ. Since S_μ maps periodically onto $[0, 1)$, we can view the phase space of S_μ as $\mathfrak{M}$ = a circle. Thus, there is no spatial boundary to worry about. By the above calculations, at $\mu = m$ where m a positive integer, the map is a dynamical $\{4m, 2m\}$ horseshoe. Therefore, there are $|\rho(4m, 2m, k)|$ regular period-k orbits at $\mu = m$. There is a unique periodic point at $\mu = 0$: a globally stable fixed point at $x = 0$, implying that $\Lambda_1 = 1$ in Theorem 1. From Theorem 2, a residual set of maps in any neighborhood of S_μ has $|\rho(4m, 2m, k)|$ solitary stem period-k cascades on $[0, m] \times \mathfrak{M}$, plus one extra stem period-one cascade, as a result of the RPO$(\mu = 0, x = 0)$. The map S_μ itself has some degenerate bifurcations, including symmetry breaking pitchfork bifurcations. However, if one permits the one-manifolds containing cascades to overlap at some isolated points, then there is no difference in counting the number of solitary cascades, and the formulas remain true.

Note that we have stated that there is exactly one extra cascade from the stable fixed point at $\mu = 0$, although the theorem only guarantees that there is a possible addition or subtraction of one cascade as a result of the RPO$(\mu = 0, x = 0)$. We are able to show this stronger statement as follows: there is an orientation of the RPO at $\mu = 0$ inherited from the index, and it is the opposite of the orientations that all that RPOs at $\mu = m > 0$ inherit from their indices. We have shown in [29] that the orientation inherited from the index remains fixed along a component Y of RPOs. Therefore, the stable fixed point at $\mu = 0$ cannot be in the same component as any RPO at $\mu = m$. It thus gives rise to an extra solitary cascade. The number of cascades for S_μ is given for two sample intervals in Table 2.

5. Cascade-free chaos and a conjecture

In this section, we give an example to show that if there is no dominant dimension of instability, then it is possible to have a route to chaos without cascades. We then conjecture that this example is demonstrating atypical

TABLE 2. The number of solitary cascades of stem period k for every large-scale perturbation of the quadratic, cubic, and quartic maps, respectively, labeled $Quad(k)$, $Cubic(k)$, $Quart(k)$, and $S_\mu(k)$. It is described in the text that for the first three maps, as long as μ_1 is small and μ_2 is sufficiently large, there are the same number of cascades. The number of cascades for S_μ is ever-growing as μ increases. We have enumerated the cascades for two sample parameter intervals here. $Quad(k)$ and $Quart(k)$ are, respectively, also equal to the number of solitary cascades of stem period k for large-scale perturbations of the Hénon map and the two-dimensional coupled quadratic system.

k	Quad(k)	Cubic(k)	Quart(k)	$S_\mu(k)$ on $[0,2]$	$S_\mu(k)$ on $[0,3]$
1	1	1	2	4+1	6+1
2	0	1	2	12	30
3	1	4	10	84	286
4	1	8	28	496	2556
5	3	24	102	3276	24,882
6	4	56	330	21756	248,534
7	9	156	1170	149,796	2,559,414
8	14	400	4064	1,048,064	26,871,264
9	28	1092	14560	7,456,512	286,654,368
$k \gg 10$	$\sim 2^k/2k$	$\sim 3^k/2k$	$\sim 4^k/2k$	$\sim 8^k/2k$	$\sim 12^k/2k$

behavior. Consider the following uncoupled one-parameter family of functions from the plane to itself:

$$f(\mu, x, y) = \mu - x^2,$$
$$g(\mu, x, y) = \mu - 5 - y^2.$$

The periodic points of this uncoupled system are of the form (a, b), where a is a periodic point for f alone, and b is a periodic point for g alone. For $\mu < -0.75$, f has no periodic points, and for $\mu > 2$, f has infinitely many periodic points, and all are unstable. For $\mu < 4.75$, g has no periodic points. At $\mu = 4.75$, a saddle-node bifurcation creates a stable and an unstable fixed point. Thus the full system has no periodic points for $\mu < 4.75$. For all $\mu > 4.75$, there are infinitely many periodic points, all of which are unstable. Thus there is chaos without cascades. The key to how this occurred is that half the regular periodic orbits have unstable dimension one, and half have unstable dimension two. Specifically, the RPOs corresponding to the stable fixed point of g have unstable dimension one, and the RPOs corresponding to the unstable fixed point of g have unstable dimension two. This lack of a dominant unstable dimension results in a route to chaos devoid of cascades.

We have shown that virtual uniform chaos is sufficient for cascades, but we conjecture that this can be weakened to asymptotically uniform chaos. This is stated more formally as follows.

Definition 5 (Asymptotically uniform chaos). Let $\rho(\mu, p, L)$ denote the set of regular periodic orbits with unstable dimension L. We say that F has asymptotically uniform chaos if there exists an N such that

$$\lim_{p\to\infty} \frac{\sum_{L\neq N} |\rho(\mu, p, L)|}{|\rho(\mu, p, N)|} = 0.$$

Conjecture 1. Assume hypotheses (A_0)–(A_3) and the weakened version of (A_4), such that at μ_2, F has asymptotically uniform chaos. Then, there are infinitely many cascades.

6. Background

In this section, we give a more detailed contrast between our work on existence and the well-known theory of scaling of cascades. We describe the scaling theory to spell out its achievements and to describe what it does not do, noting in particular that our goals are quite different. Our theory addresses the problem of describing a cascade, once it is known to exist.

The difficulty of showing that there are cascades is illustrated by the following quote from Collet and Eckmann [5], giving the problem a probabilistic interpretation:

> "If a one-parameter family shows subharmonic bifurcations to periods 1, 2, 4, for example, then there is so to speak a higher probability of finding period 8 at a further parameter variation than of finding a period 4 when only periods 1 and 2 have appeared." [5, p. 39]

That is, using these methods, one is unable to predict whether the next bifurcation of a general family of orbits will be a supercritical period doubling (the nice case) rather than an inverted saddle-node or a subcritical period-doubling bifurcation. We avoid this difficulty by use of a topological invariant that is effective for generic families of maps.

Collet and Eckmann introduce one of their goals as follows [5, p. 37].

> "Theorem. (Feigenbaum [1978], Collet-Eckmann-Lanford [1980]). For sufficiently smooth families of maps of the interval to itself the number δ does not in general depend on the family."

Of course there are extremely smooth scalar maps (like $\mu - x$ or $\cos(\mu^2 - x)$) for which there are no cascades. The assertion means that for every sufficiently smooth map, each typical cascade (if indeed there are any cascades) will scale as (1). The phrase "in general" acknowledges that there are exceptional cascades which do not satisfy this property.

There is also a local result that we quote here for maps in $\mathbb{R}^2$, but the analogue also holds for scalar families of maps. "Every one-parameter family $\mathbb{R}^2 \to \mathbb{R}^2$, which passes sufficiently close to [a universal map] will exhibit an infinite sequence of period doublings" [5, p. 57] and these will exhibit scaling. The scalar map $\mu - x^2$ is an example of a map whose first cascade is known to in fact exhibit scaling.

7. Appendix

Computing the number of regular periodic orbits for an m-shift. The following *Mathematica* function computes the first Nu values of $|\rho_s(m, j, k)|$, where $j = (m + \ell)/2$:

```
rho[m_, ell_, Nu_] := Module[{t},
  et[1] = (m + ell)/2;
  orb[1] = et[1]/1;
  SetAttributes[et, Listable];
  SetAttributes[orb, Listable];
  For[k = 2, k < Nu + 1, k++,
   J = Select[Drop[Divisors[k], -1], ! Divisible[k/#, 2] &];
   L = If[EvenQ[k], m^(k/2), 0];
   et[k] = (m^k + ell^k)/2 - Total[et[J]] - L;
   orb[k] = et[k]/k;
   ];
  t = orb[Range[Nu]]]
```

In order to see these values, we set m, ell, and Nu to specific values. For example, the line

```
rho[4, 0, 12]
```

yields the output

```
{2, 2, 10, 28, 102, 330, 1170, 4064, 14560, 52326, 190650, 698700}
```

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Madhura R. Joglekar
Department of Mathematics
University of Maryland
College Park, MD 20742, USA
e-mail: `madhura@math.umd.edu`

Evelyn Sander
Department of Mathematical Sciences
George Mason University
Fairfax, VA 22030, USA
e-mail: `esander@gmu.edu`

James A. Yorke
Department of Mathematics, IPST, and Physics Department
University of Maryland
College Park, MD 20742, USA
e-mail: `yorke@umd.edu`