Discrete and Continuous Dynamical Systems - Series B
Vol. 29, No. 2, February 2024, pp. 666-701
doi:10.3934/dcdsb.2023109

CYCLIC SYMMETRY INDUCED PITCHFORK BIFURCATIONS IN THE DIBLOCK COPOLYMER MODEL

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(Communicated by Sebastian M. Wieczorek)

Abstract. The Ohta-Kawasaki model for diblock copolymers exhibits a rich equilibrium bifurcation structure. Even on one-dimensional base domains the bifurcation set is characterized by high levels of multi-stability and numerous secondary bifurcation points. Many of these bifurcations are of pitchfork type. In previous work, the authors showed that if pitchfork bifurcations are induced by a simple $\mathbb{Z}_2$ symmetry-breaking, then computer-assisted proof techniques can be used to rigorously validate them using extended systems. However, many diblock copolymer pitchfork bifurcations cannot be treated in this way. In the present paper, we show that in these more involved cases, a cyclic group action is responsible for their existence, based on cyclic groups of even order. We present theoretical results establishing such bifurcation points and show that they can be characterized as nondegenerate solutions of a suitable extended nonlinear system. Using the latter characterization, we also demonstrate that computer-assisted proof techniques can be used to validate such bifurcations. While the methods proposed in this paper are only applied to the diblock copolymer model, we expect that they will also apply to other parabolic partial differential equations.

1. **Introduction.** Symmetry-breaking pitchfork bifurcations are a common feature of nonlinear partial differential equation models as they vary with respect to parameters. We focus here on pitchfork bifurcations of the one-dimensional Ohta–Kawasaki model for the formation of diblock copolymers [18]. In a previous paper [13], we developed a rigorous computer-assisted proof method for the validation of symmetry-breaking pitchfork bifurcations in the case of $\mathbb{Z}_2$-symmetries that were observed in [11]. These results were based on creating a validated version of the numerical methods of Werner and Spence [30], by reformulating the existence of the bifurcation point as the existence of a nondegenerate solution of an extended nonlinear system.

However, there are cases of pitchfork bifurcations that we observed in [11, 13], but were unable to validate using the above theoretical methods — as although

2020 *Mathematics Subject Classification.* Primary: 37G40, 37M20, 65G20, 65P30; Secondary: 37B35, 37C81, 65G30, 74G60, 74N15.

Key words and phrases. Bifurcations, Ohta-Kawasaki model, symmetry-breaking, pitchfork bifurcations, cyclic group, computer-assisted proofs, interval arithmetic, rigorous validation.

The research of E.S. and T.W. was partially supported by the Simons Foundation under Awards 636383 and 581334, respectively.

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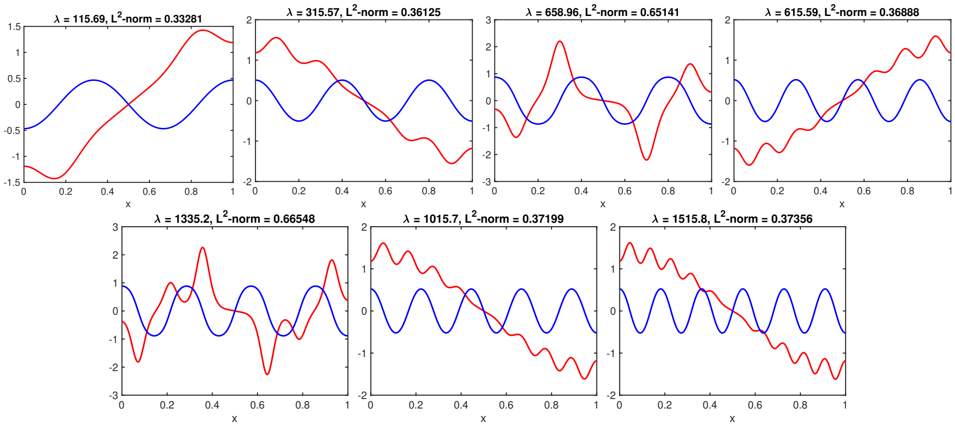

FIGURE 1. Examples of odd solutions, shown in blue, with odd eigenfunctions, depicted in red, which correspond to eigenvalue zero at the bifurcation point, where odd is measured with respect to the domain midpoint 1/2. The solutions u are n-layer solutions, equivariant under the cyclic symmetry (2), where n is 3, 5, 5, 7 in the top row, and 7, 9, 11 in the bottom row, respectively. Since both the bifurcating solution and the eigenfunction are odd, the solutions remain odd as they undergo a pitchfork bifurcation, but bifurcation breaks the cyclic symmetry.

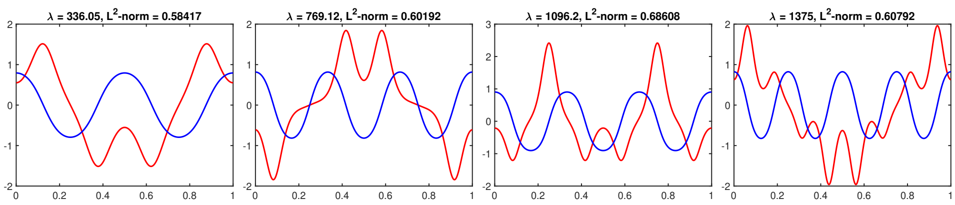

FIGURE 2. Examples of even solutions, shown in blue, with even eigenfunctions, depicted in red, which correspond to eigenvalue zero at the bifurcation point, where even is measured with respect to the domain midpoint 1/2. In each figure, the solutions u are n-layer solutions, equivariant under the cyclic symmetry (2), where n is 4, 6, 6, and 8, respectively. Since both the bifurcating solution and the eigenfunction are even, solutions remain even as they undergo a pitchfork bifurcation, but the bifurcation breaks the cyclic symmetry.

a high degree of symmetry is broken at bifurcation, all local solutions remain in the same $\mathbb{Z}_2$-symmetry class, i.e., solutions are all even or all odd with respect to the center of the one-dimensional domain. In this paper, we adapt the techniques used for the $\mathbb{Z}_2$-symmetry case in order to give theoretical underpinnings needed for rigorous computational validation of symmetry-breaking bifurcations for more general cyclic group symmetries, as shown in Figures 1 and 2 and described in more detail below.

The Ohta-Kawasaki equation is a model for diblock copolymers, materials formed by two linear polymers (known as blocks) which contain different monomers. If the blocks are thermodynamically incompatible, then the blocks try to separate after the reaction. However, since they are covalently bonded, such a separation is impossible on the macroscopic scale. This competition of long range and short range forces causes microphase separation, resulting in pattern formation. The Ohta-Kawasaki equation on a domain $\Omega \subset \mathbb{R}^d$ is given by

$$\begin{aligned} w_t &= -\Delta(\Delta w + \lambda f(w)) - \lambda\sigma(w - \mu) \quad \text{in } \Omega \,, \\ \frac{\partial w}{\partial \nu} &= \frac{\partial(\Delta w)}{\partial \nu} = 0 \quad \text{on } \partial\Omega \,, \end{aligned}$$

where ν denotes the unit outward normal on the boundary of Ω, corresponding to homogeneous Neumann boundary conditions. The quantity $w(t, x)$ is the local density difference of the two monomer blocks. That is, if $w(t, x) = -1$, then at time t and locally near the point x, the material consists entirely of block A. If instead we have $w(t, x) = 1$, then the local average of the material consists entirely of block B. For values $-1 < w(t, x) < 1$ the local material contains a mix of blocks A and B. The parameter μ is the space average of w, meaning it is a measure of the relative total proportion of the two polymers, which we tersely refer to as the *mass* of the system. The equation obeys a mass conservation, implying that μ is time-invariant. A large value of the parameter λ corresponds to a large short-range repulsion, while a large value of the parameter σ corresponds to large long-range elasticity forces. We refer the reader to [11] for a detailed description of how λ and σ are defined. See also [26] for a description of the phase separation aspects of the model. Finally, note that the second boundary condition is necessary since this is a fourth order equation. In this paper, we focus on equilibrium solutions $w = w(x)$.

For notational convenience, we reformulate our equation slightly. For a solution w of the diblock copolymer equation, we define $u = w - \mu$. Since the space average of w is μ, the average of the shifted function u is zero. Therefore the equilibrium equation becomes

$$\begin{aligned} -\Delta(\Delta u + \lambda f(u + \mu)) - \lambda\sigma u &= 0 \quad \text{in } \Omega \,, \\ \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u)}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \,, \\ \int_\Omega u \, dx &= 0 \,. \end{aligned} \tag{1}$$

We will consider this version of the equation for the rest of the paper, and restrict our attention to the case of the one-dimensional domain $\Omega = (0, 1)$ with $\mu = 0$, where $\sigma > 0$ denotes a fixed constant, and the nonlinearity is chosen as $f(u) = u - u^3$. Note that while this particular form of the nonlinearity is not critical for our results, the fact that the nonlinearity is odd plays a large role for the results of this paper. We would like to point out, however, that this oddness condition was chosen purely to simplify our presentation. One could in fact obtain similar results without it.

Figure 3 shows a numerically computed bifurcation diagram for (1) with $\sigma = 6$. The bifurcation diagram is restricted to the primary equilibrium branches emanating from the spatially homogeneous trivial solution, along with the secondary bifurcation points shown as red dots. Secondary branches do emanate from each of these branches, but they have been omitted for the sake of clarity. Some of the

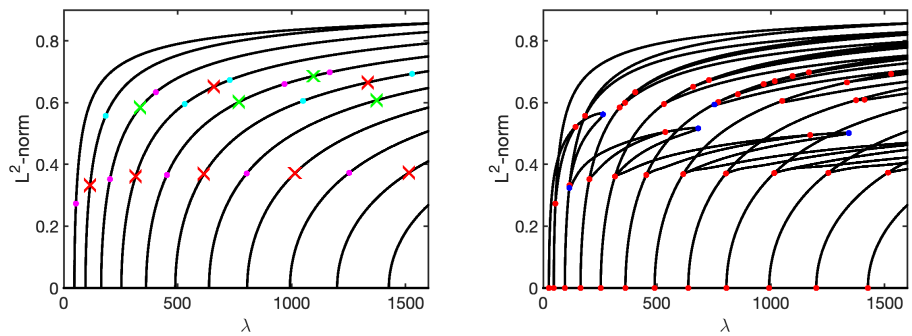

FIGURE 3. Partial bifurcation diagram for the one-dimensional diblock copolymer equation for the parameter values $\mu = 0$ and $\sigma = 6$. In the left panel, each of the dots and crosses is a bifurcation point. The red crosses correspond to the odd solutions with odd eigenfunctions shown in Figure 1, while the green crosses correspond to the even solutions with even eigenfunctions shown in Figure 2. In contrast, the cyan dots represent odd solutions with even eigenfunctions, while the magenta dots correspond to even solutions with odd eigenfunctions. These last two types are $\mathbb{Z}_2$-symmetry breaking bifurcations points. Altogether, the colored points depict all detected bifurcation points along primary branches. Originating at each bifurcation point, there are secondary branches which are omitted for the sake of clarity. They are included, however, in the right panel, which illustrates that the branches are connected through multiple routes.

depicted bifurcation points are $\mathbb{Z}_2$-symmetry breaking, as covered in [13]. However, in Figures 1 and 2 we show that for nine cases, there is no $\mathbb{Z}_2$-symmetry broken at the pitchfork bifurcation. Instead, these bifurcation solutions are n-layer solutions which are equivariant under the following cyclic symmetry.

Suppose that u denotes the solution at one of these bifurcation points, and let ϕ denote the eigenfunction of the Fréchet derivative of (1) at u corresponding to the eigenvalue 0, which we further assume to be simple. Furthermore, suppose that we have extended the solution u from $\Omega = (0, 1)$ to all of $\mathbb{R}$ via successive even reflections across the boundary. Then each such solution u satisfies the cyclic symmetry given by

$$(T_n u)(x) \;=\; -u\left(x + \frac{1}{n}\right) \;=\; u(x) \qquad \text{for all} \qquad x \in \mathbb{R}\,, \tag{2}$$

for some $n \in \mathbb{N}$. In contrast, the eigenfunction ϕ does not display this type of symmetry, in fact, it seems to have no special symmetry properties at all. We will see later that the operator T_n is the generator of a cyclic group, but since its natural functional-analytic domain interferes with our homogeneous Neumann boundary conditions, we defer precise statements about the spaces on which T_n is defined and the order of the generated cyclic group until the next section.

In this paper, we develop the theoretical foundation for a rigorous computer-assisted proof method for showing that the functions shown in Figures 1 and 2 do indeed give rise to symmetry-breaking pitchfork bifurcations. This is accomplished by first establishing a mathematical bifurcation result on pitchfork bifurcations induced by a cyclic group action, and then equivalently reformulating it as a zero-finding problem for a suitable extended nonlinear system, in the spirit of [13]. The latter system can then in principle be solved using computer-assisted proofs based on the constructive implicit function theorem introduced in [23, 27]. Although in this paper we concentrate on the Ohta-Kawasaki equation, our methods can be adapted to a much more general set of equations. In particular, our pitchfork bifurcation result is quite general, in that along with technical assumptions, it only relies on being able to divide the space into the direct sum of pairwise orthogonal spaces which exhibit certain invariance properties. One of these spaces contains the function at bifurcation, and another contains the eigenfunction spanning the associated one-dimensional kernel of the operator. The result does not include the specific form (2) of the symmetry in the statement, but in our application, these pairwise orthogonal spaces come from symmetry considerations. Note that while the solution-eigenfunction pairs shown in Figures 1 and 2 appear to be orthogonal in the sense that $\int_0^1 u(x)\phi(x)\,dx = 0$, there is no immediately obvious symmetry which would force this identity. In fact, it will be shown later that despite the lack of any obvious symmetry, each eigenfunction ϕ exhibits a more subtle one, which is far from obvious and which is responsible for this orthogonality.

A number of papers have previously considered numerical computation of bifurcation diagrams for the Ohta-Kawasaki and Cahn-Hilliard equations, such as for example [4, 8, 11, 14]. There are also several decades of results on computer validation for dynamical systems and differential equations solutions which combine fixed point arguments and interval arithmetic, see for example [2, 9, 17, 19, 22, 31]. There are several papers that have already considered rigorous validation of parameter-dependent solutions for the Ohta-Kawasaki model [3, 24, 25], as well as in other contexts [7, 12, 15]. However, this is the first study to look at computer-assisted proofs of higher symmetry-breaking bifurcations for the Ohta-Kawasaki model.

The remainder of this paper is organized as follows. In Section 2 we describe the symmetry spaces associated with the cyclic group action given by (2), discuss their essential properties, and explain why the underlying symmetry group responsible for the bifurcation is in fact $\mathbb{Z}_{2n}$. In Section 3, we state and prove the analytical $\mathbb{Z}_{2n}$-equivariant pitchfork bifurcation result. Furthermore, by reformulating the problem as a zero-finding problem for an extended system, we also are able to establish computationally testable conditions. We would like to point out that while the results of this section are formulated only for the specific situation considered in this paper, the general approach should be applicable in much more generality. This is described in more detail in the context of Remark 3.4, which collects the essential assumptions that are necessary. In Section 4, we introduce the computational validation methods required, based on recent results from [20]. This paper is primarily focused on the analysis of this new type of symmetry-breaking pitchfork bifurcations, but for proof of concept, we end the paper with sample solution validations of pitchfork bifurcation points from Figures 1 and 2. Nevertheless, a few computational challenges remain, and we briefly address these as well as potential solution attempts.

2. **Cyclic equivariance of diblock copolymers.**

In this section we describe the equivariance properties of the equilibrium diblock copolymer model with respect to the cyclic symmetry mentioned in the introduction. We start in Section 2.1 by presenting our basic functional-analytic setup. After briefly discussing the difficulties with using the cyclic symmetry defined in (2) directly in this framework, Section 2.2 shows that these issues can be overcome by considering a detour through larger function spaces. In combination with additional symmetries, one can then in fact use the symmetry operator T_n to derive a suitable symmetry-induced decomposition of the original spaces. We also present orthogonality and invariance properties which are essential for our application. Finally, Section 2.3 establishes necessary equivariance properties of the nonlinear diblock copolymer operator.

2.1. **Basic functional-analytic setup.** We begin by briefly presenting the framework for our study of the diblock copolymer model, which had already been used in our previous work [11, 13, 24, 27]. As mentioned in the introduction, we restrict consideration to equilibria for the Ohta-Kawasaki model given in (1) on the one-dimensional domain $\Omega = (0,1)$, and study only the zero mass case $\mu = 0$. In addition, the parameter σ is fixed and strictly positive, and we use the cubic odd nonlinearity $f(u) = u - u^3$. From a functional-analytic point of view, we can then rearrange the system of three equations in such a way that the equilibrium solutions are zeros of a single nonlinear operator F, while the boundary and integral conditions are absorbed into the definition of the operator domain. In this way, equilibria of the problem (1) correspond to solutions of the zero finding problem

$$F(\lambda, u) = -\Delta(\Delta u + \lambda f(u + \mu)) - \lambda\sigma u = 0 \,, \tag{3}$$

where we have $F : \mathbb{R} \times X \to Y$ with respect to the spaces

$$X = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \text{ and } \int_\Omega u\,dx = 0 \right\} \quad \text{and} \quad Y = H^{-2}(\Omega) \,. \tag{4}$$

These spaces are both Hilbert spaces, equipped for our purposes with the norms

$$\|u\|_X = \|\Delta u\|_{L^2(\Omega)} \qquad \text{and} \qquad \|u\|_Y = \left\|\Delta^{-1} u\right\|_{L^2(\Omega)} \,,$$

where one can verify that the mapping $\Delta : L^2(\Omega) \cap \left\{ \int_\Omega u\,dx = 0 \right\} \to H^{-2}(\Omega)$ is an isometry. Standard results imply that in this setting the operator F is a well-defined smooth operator. Furthermore, since we assumed the identity $\mu = 0$ and f is an odd function, we also have $F(\lambda, -u) = -F(\lambda, u)$ for all $\lambda \in \mathbb{R}$ and $u \in X$.

Of particular importance for the detection of bifurcation points is of course the Fréchet derivative of F at a given equilibrium solution. Thus, in the following, we consider a fixed parameter value $\lambda_0 \in \mathbb{R}$ and a function $u_0 \in X$, and we let L denote the Fréchet derivative of F at the pair (λ_0, u_0) given by

$$L[v] \;=\; D_u F(\lambda_0, u_0)[v] \;=\; -\Delta(\Delta v + \lambda f'(u_0 + \mu) v) - \lambda\sigma v \,. \tag{5}$$

According to its definition, one has $L = D_u F(\lambda_0, u_0) \in \mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y, equipped with the operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. With the range and null space of this linear operator we associate the following orthogonal projections.

Definition 2.1 (Orthogonal projections P and Q associated with L)**.** Let L be the Fréchet derivative of the diblock copolymer operator as defined in (5). Then we denote by $Q : X \to X$ the orthogonal projection of the domain X onto the null space $N(L)$, and we let $P : Y \to Y$ be the orthogonal projection of Y onto the orthogonal complement of the range $R(L)$. In other words, we define the closed subspaces $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$ such that

$$X = N(L) \oplus \tilde{X} \quad \text{and} \quad Y = \tilde{Y} \oplus R(L) , \qquad \text{where} \quad \tilde{X} \perp N(L) \quad \text{and} \quad \tilde{Y} \perp R(L) .$$

Thus, the projector $P : Y \to Y$ is characterized by $R(P) = \tilde{Y}$ and $N(P) = R(L)$, while the projection $Q : X \to X$ satisfies both $R(Q) = N(L)$ and $N(Q) = \tilde{X}$.

These projection operators allow us to apply standard arguments based on the Lyapunov-Schmidt reduction to establish pitchfork bifurcations induced by the action of a cyclic group.

Finally, we impose the following two assumptions on the operator F and the pair (λ_0, u_0), which are standard for the discussion of bifurcation points. The first of these has been verified for the diblock copolymer operator in [13], and the second one will be satisfied at all pitchfork bifurcation points shown in Figures 1 and 2.

Hypothesis 2.2 (Fredholm property)**.** Assume that the operator $F : \mathbb{R} \times X \to Y$ is nonlinear and sufficiently smooth, and suppose that the pair $(\lambda_0, u_0) \in \mathbb{R} \times X$ is a zero of the operator F, i.e., we assume that it satisfies the identity $F(\lambda_0, u_0) = 0$. Furthermore, suppose that the Fréchet derivative $L = D_u F(\lambda_0, u_0)$ of F at (λ_0, u_0) is a Fredholm operator of index zero.

Hypothesis 2.3 (One-dimensional kernel)**.** Suppose that the above Fredholm Hypothesis 2.2 is satisfied. In addition, assume that the Fréchet derivative L has a one-dimensional null space. Since L is of index zero, this immediately implies that its range has codimension one. Therefore, there exist two nonzero elements $\phi_0 \in X$ and $\psi_0^* \in Y^*$ such that both

$$N(L) = \operatorname{span}(\phi_0) \qquad \text{and} \qquad R(L) = N(\psi_0^*)$$

are satisfied. Furthermore, these assumptions show that the projections P and Q from Definition 2.1 both have rank one.

2.2. **Space decompositions induced by cyclic symmetry.** We now turn our attention to studying the cyclic symmetry operator T_n defined in (2). As we mentioned in the introduction, all of the pitchfork bifurcation equilibria shown in Figures 1 and 2 are fixed points of this operator. Note that since the definition of T_n includes a shifted argument, we had to extend the definition of the underlying functions beyond the bounded domain $\Omega = (0, 1)$ by even reflections. More precisely, consider for the moment an arbitrary function $u \in X$, where the space X was defined in (4) above. In view of the imposed homogeneous Neumann boundary conditions, we can extend the function u smoothly to a periodic function $\tilde{u}$ on $\mathbb{R}$, by first defining

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for} \quad 0 \le x \le 1 , \\ u(2 - x) & \text{for} \quad 1 < x < 2 , \end{cases}$$

and then using the identity $\tilde{u}(x + 2k) = \tilde{u}(x)$ for all $x \in [0, 2]$ and $k \in \mathbb{Z}$. In the following, we will refer to $u \in X$ and $\tilde{u} : \mathbb{R} \to \mathbb{R}$ as *corresponding functions*, or equivalently, we will say that $\tilde{u}$ *is the extension of* u.

With the notion of corresponding functions, it now makes sense to apply the symmetry operator T_n. One can immediately see that for the equilibrium-eigenfunction pairs shown in Figures 1 and 2, the extension $\tilde{u}$ of every solution u is indeed a fixed point of the operator T_n, where n denotes the number of layers of u. On the other hand, extensions $\tilde{\phi}$ of the eigenfunctions ϕ are not fixed points of T_n.

In view of these observations, it seems plausible to expect that standard approaches to studying symmetry-induced bifurcations should apply directly in our situation, such as the ones described in [5, 10]. Notice, however, that for these approaches to work one needs to study symmetry operators which are *acting on the space containing the equilibrium solutions* — and in our case this is the Hilbert space X. Yet, one can easily see that if ϕ is one of the eigenfunctions in Figures 1 or 2, then the function $T_n(\tilde{\phi})$ no longer satisfies homogeneous Neumann boundary conditions on Ω. In other words, *the restriction of* $T_n(\tilde{\phi})$ *to* Ω *is no longer an element of* X. In addition, even in situations where one can use abstract equivariant bifurcation theory, one still has to verify the actual bifurcation type and the corresponding nondegeneracy conditions in the specific underlying system, as explained in detail in [16].

At first glance, this observation appears to doom the use of the symmetry operator T_n. Nevertheless, we will show in the remainder of this subsection that this is far from the truth. In fact, we will be able to study T_n on a larger Hilbert space which contains X, but on which the action of T_n is well-defined — and then use the obtained insight to construct an appropriate space decomposition of X.

To introduce this larger space, we first return to the definition of the extended function $\tilde{u} : \mathbb{R} \to \mathbb{R}$ corresponding to an element $u \in X$. Notice that according to our construction this extension satisfies $\tilde{u}(x + 2) = \tilde{u}(x)$ for all $x \in \mathbb{R}$. Thus, its restriction to $(0, 2)$ automatically satisfies the periodic boundary conditions $\tilde{u}(0) = \tilde{u}(2)$ and $\tilde{u}'(0) = \tilde{u}'(2) = 0$. Furthermore, one can immediately see that the symmetry operator T_n defined in (2) maps every 2-periodic function to another 2-periodic function.

With this in mind, we introduce three spaces of 2-periodic functions, two of which will extend our Hilbert spaces X and Y. More precisely, we consider

$$\begin{aligned} H^2_{per}(\mathbb{R}) &= \left\{ v \in H^2_{loc}(\mathbb{R}) \;:\; v(x+2) = v(x) \;\text{ for }\; x \in \mathbb{R}\,, \;\int_0^2 v\,dx = 0 \right\}, \\ L^2_{per}(\mathbb{R}) &= \left\{ v \in L^2_{loc}(\mathbb{R}) \;:\; v(x+2) = v(x) \;\text{ for }\; x \in \mathbb{R}\,, \;\int_0^2 v\,dx = 0 \right\}, \quad (6) \\ H^{-2}_{per}(\mathbb{R}) &= \Delta L^2_{per}(\mathbb{R}), \end{aligned}$$

where the space $L^2_{loc}(\mathbb{R})$ denotes the space of all measurable real-valued functions on $\mathbb{R}$ which are square integrable on compact intervals, and $H^2_{loc}(\mathbb{R}) \subset L^2_{loc}(\mathbb{R})$ the space of all twice weakly differentiable Sobolev functions whose first two derivatives are in $L^2_{loc}(\mathbb{R})$ as well. All three of the above spaces are Hilbert spaces with respect to the norms

$$\begin{aligned} \|v\|_2 &= \|\Delta v\|_{L^2(0,2)} && \text{for} \quad v \in H^2_{per}(\mathbb{R})\,, \\ \|v\|_0 &= \|v\|_{L^2(0,2)} && \text{for} \quad v \in L^2_{per}(\mathbb{R})\,, \\ \|v\|_{-2} &= \|\Delta^{-1} v\|_{L^2(0,2)} && \text{for} \quad v \in H^{-2}_{per}(\mathbb{R})\,, \end{aligned}$$

respectively. Since we have restricted ourselves to functions with mean zero, one can verify that both mappings $\Delta : H^2_{per}(\mathbb{R}) \to L^2_{per}(\mathbb{R})$ and $\Delta : L^2_{per}(\mathbb{R}) \to H^{-2}_{per}(\mathbb{R})$ are isometries.

Our interest in these spaces is two-fold. On the one hand, they are spaces of periodic functions which in some sense contain our fundamental Hilbert spaces X

and Y. To see this, note that for every function $u \in X$, its extension clearly satisfies $\tilde{u} \in H^2_{per}(\mathbb{R})$. However, it is not true in general that the restriction of every function in the latter space lies in X. Nevertheless, for any $u \in X$ and $k \in \mathbb{Z}$, the construction of its corresponding function $\tilde{u}$ implies

$$\tilde{u}(-x) = \tilde{u}(\underbrace{-x+2k}_{\in[0,2]}) = \tilde{u}(2-(-x+2k)) = \tilde{u}(x+2(1-k)) = \tilde{u}(x)$$

for all $x \in \mathbb{R}$, i.e., the extension is an even function. Conversely, one can see that every even function $v \in H^2_{per}(\mathbb{R})$ satisfies $v(1+x) = v(-(1+x)) = v(2-(1+x)) = v(1-x)$, i.e., it is also even with respect to $x = 1$. Since every function in $H^2_{per}(\mathbb{R})$ is continuously differentiable in view of Sobolev's embedding theorem [1], this in turn implies that the restriction of any even function $v \in H^2_{per}(\mathbb{R})$ to Ω satisfies homogeneous Neumann boundary conditions, and we have shown that in fact

$$X = \left\{ v|_\Omega \;:\; v \in H^2_{per}(\mathbb{R}) \;\text{ and }\; v(x) = v(-x) \;\text{ for all }\; x \in \mathbb{R} \right\} . \tag{7}$$

Our second interest in the above spaces stems from the fact that they are invariant under the symmetry operator defined in (2). More precisely, let W denote either $H^2_{per}(\mathbb{R})$ or $L^2_{per}(\mathbb{R})$. Then we can clearly define an isometry $T_n : W \to W$ via

$$T_n(v)(x) = -v\left(x + \frac{1}{n}\right) .$$

In addition, for $W = H^{-2}_{per}(\mathbb{R})$ one can set $T_n(v) = \Delta T_n(\Delta^{-1} v)$. In other words, the symmetry operator T_n is a well-defined action on these spaces.

As we stated above, we will use the operator T_n on the spaces of periodic functions to ultimately introduce a decomposition of the spaces X and Y from the last subsection. For this, however, we need to first study T_n on the former spaces. In the following, we begin by considering the cases $W = H^2_{per}(\mathbb{R})$ and $W = L^2_{per}(\mathbb{R})$, since in these cases the elements are actually functions that can be evaluated pointwise. The Sobolev space with negative exponent will be treated subsequently.

It is immediately clear that the operator $T_n : W \to W$ has the property of being cyclic of order $2n$, and that it commutes with the Laplacian, i.e., we have both

$$T_n^{2n} = I \qquad \text{and} \qquad \Delta T_n = T_n \Delta .$$

The first property in particular implies that the minimal polynomial for T_n is given by

$$m(t) \;=\; t^{2n} - 1 \;=\; (t^n - 1)(t^n + 1) \;=\; \underbrace{(t-1)}_{m_a(t)} \underbrace{(t^{n-1} + t^{n-2} + \cdots + 1)}_{m_b(t)} \underbrace{(t^n + 1)}_{m_c(t)} .$$

In addition, one can easily verify that $(t^n + 1)/2 - (t^n - 1)/2 = 1$ and that 1 is not a root of either m_b or m_c. This in turn implies that the three polynomials m_a, m_b, and m_c are relatively prime, and therefore we have the decomposition

$$\begin{aligned} W \;&=\; N(T_n - I) \;\oplus\; N(T_n^{n-1} + T_n^{n-2} + \cdots + I) \;\oplus\; N(T_n^n + I) \\ &=\; \underbrace{N(m_a(T_n))}_{W_a} \;\oplus\; \underbrace{N(m_b(T_n))}_{W_b} \;\oplus\; \underbrace{N(m_c(T_n))}_{W_c} . \end{aligned} \tag{8}$$

Each of these three subspaces has additional important properties which are crucial for our applications, and which will be studied in more detail below. For now, we would like to point out that the space W_a consists of functions v which satisfy the identity $v(x + 1/n) = -v(x)$ for all $x \in \mathbb{R}$. Thus, by inspection, one would suspect that for the diblock copolymer equilibrium solutions u shown in Figures 1 and 2, their respective corresponding functions $\tilde{u}$ lie in W_a for $W = H^2_{per}(\mathbb{R})$, if n denotes the number of layers of u. As we will see later, the respective eigenfunctions will automatically be contained in one of the remaining two subspaces.

The decomposition of the space W into a direct sum of three subspaces lies at the heart of our approach, and we will now show that this decomposition can be pulled down to the subspace X. We have already seen in (7) that the space X occurs naturally as a subspace of $W = H^2_{per}(\mathbb{R})$ if we additionally impose an evenness constraint. As the following result shows, this latter constraint plays well with the decomposition $W = W_a \oplus W_b \oplus W_c$.

Lemma 2.4 (Invariance under reflection). *Let $W = H^2_{per}(\mathbb{R})$ or $W = L^2_{per}(\mathbb{R})$, and suppose that $u \in W$ is arbitrary. Furthermore, suppose that $v \in W$ is defined via $v(x) = u(-x)$ for all $x \in \mathbb{R}$. Then for every $\tau \in \{a, b, c\}$ one has the implication*

$$u \in W_\tau \qquad \Longrightarrow \qquad v \in W_\tau \quad \text{and} \quad u + v \in W_\tau \; .$$

Proof. Notice first that we only have to establish the validity of $v \in W_\tau$ in the above implication. According to its definition, the space W_τ is a linear subspace, and therefore the inclusions $u, v \in W_\tau$ immediately imply $u + v \in W_\tau$ as well.

Now let u and v be given as in the formulation of the lemma. Then the periodicity of u implies $v(x) = u(-x) = u(2 - x)$, and this in turn yields

$$T_n^k v(x) = (-1)^k v\left(x + \frac{k}{n}\right) = (-1)^k u\left(2 - x - \frac{k}{n}\right) \tag{9}$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$. We now distinguish between the three cases $\tau \in \{a, b, c\}$.

To begin with, let $u \in W_a$. Then we have $u(t) = T_n u(t) = -u(t + 1/n)$, and this readily implies $u(t) = -u(t - 1/n)$. If one now substitutes $t = 2 - x$, then (9) gives

$$T_n v(x) = -u\left(2 - x - \frac{1}{n}\right) = u(2 - x) = u(-x) = v(x) \; ,$$

i.e., we also have $v \in W_a$.

Consider now the case $u \in W_b$. Then the equation $\sum_{k=0}^{n-1} T_n^k u(s) = 0$ holds for all $s \in \mathbb{R}$. Therefore, if we set $s = 2 - x - 1 + 1/n$, then one obtains with (9) the identity

$$\begin{aligned}
\sum_{k=0}^{n-1} T_n^k v(x) &= \sum_{k=0}^{n-1} (-1)^k u\left(2 - x - \frac{k}{n}\right) \\
&= \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} u\left(2 - x - \frac{n-1-\ell}{n}\right) \\
&= (-1)^{n-1} \sum_{\ell=0}^{n-1} (-1)^\ell u\left(s + \frac{\ell}{n}\right) \; = \; (-1)^{n-1} \sum_{\ell=0}^{n-1} T_n^\ell u(s) \; = \; 0 \; ,
\end{aligned}$$

where the second equality uses the index change $\ell = n - 1 - k$, and for the third one we note that $(-1)^\ell = (-1)^{-\ell}$. This shows that $v \in W_b$.

Finally, let us assume that $u \in W_c$. Then $-u(s) = T_n^n u(s) = (-1)^n u(s + 1)$ for all $s \in \mathbb{R}$. Thus, if we set $s = -x$, then we obtain

$$T_n^n v(x) \;=\; (-1)^n v\,(x + 1) = (-1)^n u\,(2 - x - 1) = (-1)^n u(-x + 1)$$

$$= \quad -u(-x) = -v(x) \, ,$$

which in turn implies $v \in W_c$. This completes the proof of the lemma. □

Remark 2.5 (Dihedral group D_{2n} action)**.** While our main focus so far has been to understand how the action of the cyclic group $\mathbb{Z}_{2n}$ induced by T_n on $W = H^2_{per}(\mathbb{R})$ or $W = L^2_{per}(\mathbb{R})$ can be used to find a suitable space decomposition, Lemma 2.4 illustrates another point. In addition to the action of T_n, our study also makes crucial use of the reflection symmetry $u(\cdot) \mapsto u(-\cdot)$ on the space W. Thus, the actual underlying symmetry group is the resulting dihedral group D_{2n} acting on W.

The above result shows that the spaces in the decomposition of W are invariant under the reflection $x \mapsto -x$. This leads us immediately to the following result, which further decomposes every W_τ into even and odd functions.

Lemma 2.6 (Even and odd decomposition)**.** *Let $W = H^2_{per}(\mathbb{R})$ or $W = L^2_{per}(\mathbb{R})$, and define the subspaces $W^e = \{u \in W : u \text{ is even}\}$ and $W^o = \{u \in W : u \text{ is odd}\}$ of W consisting of all even and odd functions, respectively. Then we have the equality $W = W^e \oplus W^o$, as well as*

$$W_\tau = (W_\tau \cap W^e) \oplus (W_\tau \cap W^o) \qquad \text{for all} \qquad \tau \in \{a, b, c\} \, .$$

Finally, functions in W^e are even with respect to both $x = 0$ and with respect to $x = 1$, while functions in W^o are odd with respect to both of these x-values.

Proof. It is well-known that every function u defined on $\mathbb{R}$ can be written as the sum of an even and an odd function in the form $u = u^e + u^o$, where the even and odd parts are explicitly given by $u^e(x) = (u(x) + u(-x))/2$ and $u^o(x) = (u(x) - u(-x))/2$, respectively. Thus, in view of Lemma 2.4 we have both $u^e \in W_\tau$ and $u^o \in W_\tau$, as long as $u \in W_\tau$. Since only the zero function is both even and odd, this implies the decompositions stated in the lemma. The statement concerning the evenness of every $u \in W^e$ with respect to $x = 1$ has already been shown in the verification of (7). Finally, for $v \in W^o$ one obtains

$$v(1-x) = -v(x-1) = -v(x-1+2) = -v(1+x) \quad \text{ for all } \quad x \in \mathbb{R} \, ,$$

and this completes the proof of the lemma. □

As we already showed in (7), the space X defined in (4) can be considered as a subspace of $W = H^2_{per}(\mathbb{R})$ in the sense that $u \in X$ if and only if its extension $\tilde{u}$ is an even function in $H^2_{per}(\mathbb{R})$. Thus, the above lemma allows us to pull the space decomposition defined in (8) down to the space X, by considering only the even functions in the spaces W_a, W_b, and W_c. More precisely, we have the following definition.

Definition 2.7 (Symmetry induced space decomposition of X)**.** Let $W = H^2_{per}(\mathbb{R})$ denote the space defined in (6), and let X be defined as in (4). Then we define three subspaces of X by considering only the even corresponding functions in the subspaces W_a, W_b, and W_c defined in (8) for some integer $n \in \mathbb{N}$, i.e., we set

$$X_\tau = \{u \in X \;:\; \tilde{u} \in W_\tau\} \qquad \text{for all} \qquad \tau \in \{a, b, c\}$$

in view of (7). Notice that Lemma 2.6 immediately implies $X = X_a \oplus X_b \oplus X_c$.

	$v \in X_a \oplus X_b$	$v \in X_c$
n even	v even with respect to $x = 1/2$	v odd with respect to $x = 1/2$
n odd	v odd with respect to $x = 1/2$	v even with respect to $x = 1/2$

TABLE 1. Additional symmetries of functions in the spaces X_a, X_b, and X_c introduced in Definition 2.7. Depending on whether the underlying integer $n \in \mathbb{N}$ is even or odd, functions in the spaces $X_a \oplus X_b$ and X_c have an additional even or odd symmetry with respect to the center point $x = 1/2$ of the domain $\Omega = (0, 1)$, as listed in the above table.

With the above definition we have achieved our first goal, namely, the derivation of a decomposition of our domain X that is in some sense induced by the symmetry T_n, and that allows us to discuss symmetry-breaking pitchfork bifurcations. Based on our derivation, one would suspect that the pitchfork bifurcation equilibria shown in Figures 1 or 2 are contained in the spaces X_a — and we still need to understand why the spaces X_b and X_c are the correct spaces to include the eigenfunctions. We would like to emphasize one more time, however, that while in some sense X_a is invariant under the symmetry T_n (via corresponding functions), this is not true for the spaces X_b and X_c.

In addition to the symmetry properties discussed so far, the functions in the spaces X_a, X_b, and X_c introduced in Definition 2.7 exhibit one more symmetry. This is the subject of the following simple lemma.

Lemma 2.8 (Symmetry with respect to $x = 1/2$)**.** *Consider the spaces X_a, X_b, and X_c introduced in Definition 2.7. Then functions in the direct sum $X_a \oplus X_b$ are even or odd with respect to the center point $x = 1/2$ of the domain $\Omega = (0, 1)$ if the integer $n \in \mathbb{N}$ is even or odd, respectively. In addition, functions in X_c are odd or even with respect to $x = 1/2$ if n is even or odd, respectively. This is summarized in Table 1.*

Proof. Consider first the case $v \in X_a \oplus X_b$, and let $\tilde{v}$ denote its corresponding function in W, which according to Definition 2.7 and (8) is contained in $W_a \oplus W_b = N(T_n^n - I)$. Thus, the extension $\tilde{v}$ satisfies the identity $T_n^n \tilde{v} = \tilde{v}$, and by iterating the definition of T_n one can easily see that this is equivalent to the identity $(-1)^n \tilde{v}(1+x) = \tilde{v}(x)$ for all $x \in \mathbb{R}$. If we then replace x by $x - 1/2$, this immediately implies

$$(-1)^n \tilde{v}\left(\frac{1}{2} + x\right) = \tilde{v}\left(x - \frac{1}{2}\right) = \tilde{v}\left(\frac{1}{2} - x\right) \quad \text{for all} \quad x \in \mathbb{R} ,$$

where for the second identity we use the fact that $\tilde{v}$ is even. This establishes the first half of the lemma. By a completely analogous argument, using the fact that for $v \in X_c$ its extension satisfies $\tilde{v} \in W_c = N(T_n^n + I)$, one can easily verify the second half as well. All that changes is the introduction of an additional negative sign, which is responsible for the switch between even and odd in this case. □

We would like to point out explicitly that, in view of the above lemma, for any given integer $n \in \mathbb{N}$ either $X_a \oplus X_b$ contains even functions with respect to $x = 1/2$ and X_c contains odd ones, or vice versa. We will see later that this fact is inherently responsible for the $\mathbb{Z}_2$ symmetry-breaking results of [13]. As it turns out, the further decomposition into X_a and X_b allows us to treat the bifurcation points shown in Figures 1 and 2.

As our final result concerning the decomposition of the space X we now show that the spaces X_a, X_b, and X_c are pairwise orthogonal. In fact, the following result implies that they are orthogonal with respect to a variety of possible inner products on X.

Lemma 2.9 (Orthogonality of the X-decomposition)**.** *The spaces X_a, X_b, and X_c introduced in Definition 2.7 are pairwise orthogonal with respect to the inner product*

$$(\Delta u, \Delta v)_{L^2(\Omega)} + \beta(\nabla u, \nabla v)_{L^2(\Omega)} + \gamma(u,v)_{L^2(\Omega)} \qquad \text{for} \qquad u, v \in X \ ,$$

for any choice of constants $\beta \geq 0$ and $\gamma \geq 0$.

Proof. Assume that the functions u and v are taken from two different spaces of X_a, X_b, and X_c, and let $\tilde{u}$ and $\tilde{v}$ denote their extensions in W.

We begin by showing that the standard $L^2(\Omega)$-inner product of u and v vanishes. In view of Lemma 2.8, this is trivially satisfied if $u \in X_a \cup X_b$ and $v \in X_c$, or vice versa, since in these cases the product uv is always odd with respect to $x = 1/2$, and therefore $\int_0^1 u(x)v(x)\,dx = 0$. Assume therefore that $u \in X_a$ and $v \in X_b$. Then one obtains

$$\begin{aligned}
\int_0^1 u(x)v(x)\,dx &= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} u(x)v(x)dx = \sum_{k=0}^{n-1} \int_0^{\frac{1}{n}} u\left(x + \frac{k}{n}\right) v\left(x + \frac{k}{n}\right) dx \\
&= \sum_{k=0}^{n-1} \int_0^{\frac{1}{n}} \left((-1)^k u\left(x + \frac{k}{n}\right)\right) \left((-1)^k v\left(x + \frac{k}{n}\right)\right) dx \\
&= \sum_{k=0}^{n-1} \int_0^{\frac{1}{n}} \left(T_n^k \tilde{u}(x)\right) \left(T_n^k \tilde{v}(x)\right) dx = \sum_{k=0}^{n-1} \int_0^{\frac{1}{n}} \tilde{u}(x)\, T_n^k \tilde{v}(x)\,dx \\
&= \int_0^{\frac{1}{n}} u(x) \left(\sum_{k=0}^{n-1} T_n^k \tilde{v}(x)\right) dx = 0 \ ,
\end{aligned}$$

where we used the facts that $T_n^k \tilde{u} = \tilde{u}$ and $\sum_{k=0}^{n-1} T_n^k \tilde{v} = 0$.

The remaining two terms in the inner products defined in the formulation of the lemma can be discussed similarly. The statement for the second derivative term $(\Delta u, \Delta v)_{L^2(\Omega)}$ follows completely analogously, since the Laplacian Δ commutes with T_n and preserves any even or odd symmetry with respect to $x = 1/2$. Finally, by using integration by parts and the boundary conditions imposed in X, one can obtain $(\nabla u, \nabla v)_{L^2(\Omega)} = -(\Delta u, v)_{L^2(\Omega)}$, and the result follows again as before. □

With the above result we have completed the construction of a decomposition of the Hilbert space X defined in (4), which serves as the domain of our nonlinear diblock copolymer operator $F : X \to Y$ introduced in (3). We now turn our attention to the image space Y.

Definition 2.10 (Symmetry induced space decomposition of Y)**.** Consider the Hilbert space $Y = H^{-2}(\Omega)$ for $\Omega = (0,1)$ introduced in (4). Let u be any element of Y. Then its inverse image $\Delta^{-1}u$ with respect to the restricted Laplacian operator $\Delta : L^2(\Omega) \cap \left\{\int_\Omega u\,dx = 0\right\} \to H^{-2}(\Omega)$ has an extension $\hat{u} \in L^2_{per}(\mathbb{R})$, given by the formula $\hat{u} = \widetilde{\Delta^{-1}u}$. Furthermore, since the Laplacian is an isometry with respect to our chosen norms, we have with $W = L^2_{per}(\mathbb{R})$ the decomposition

$$L^2_{per}(\mathbb{R}) \; = \; W_a \oplus W_b \oplus W_c \; ,$$

where W_a, W_b, and W_c contained in $L^2_{per}(\mathbb{R})$ were defined in (8). The subspaces in this decomposition are pairwise orthogonal. Thus, if we define

$$Y_\tau = \{ u \in Y \; : \; \hat{u} \in W_\tau \} \qquad \text{for all} \qquad \tau \in \{a, b, c\}$$

then one can verify that $Y = Y_a \oplus Y_b \oplus Y_c$, and that the involved subspaces are pairwise orthogonal. In fact, one can also show that $Y_\tau = \Delta^2 X_\tau$ for all $\tau \in \{a, b, c\}$. Finally, using our earlier definition of $T_n : H^{-2}_{per}(\mathbb{R}) \to H^{-2}_{per}(\mathbb{R})$ via the identity $T_n(u) = \Delta T_n(\Delta^{-1} u)$, one can verify that we have $u \in Y_\tau$ if and only if one has the equality $m_\tau(T_n)[\Delta \hat{u}] = 0$. In the following, we will therefore call $\Delta \hat{u} \in H^{-2}_{per}(\mathbb{R})$ the extension of $u \in Y$.

The above descriptions of the spaces X and Y, and of their respective decompositions, are tailor-made for the results discussed in the remainder of the paper. Nevertheless, we close this subsection with an alternative description of these spaces, which is based on cosine Fourier series.

Remark 2.11 (Cosine Fourier series representations)**.** Consider an arbitrary function $u \in X$. Then it was shown in [24] that there exists a unique representation

$$u(x) \; = \; \sum_{k \in \mathbb{N}} a_k \cos(k \pi x) \qquad \text{with} \qquad a_k \in \mathbb{R} \quad \text{for} \quad k \in \mathbb{N} \; , \tag{10}$$

where the constant term $k = 0$ is omitted due to the zero mass constraint, and the series converges with respect to our chosen norm on X.

As it turns out, each of the three subspaces X_a, X_b, and X_c have a simple description in terms of this series representation, since the basis functions are pairwise orthogonal, and every one of these functions is contained in exactly one of these spaces.

To begin with, consider the space X_a. For any function $u \in X_a$, if we denote its extension again by $\tilde{u}$, one can easily see that in view of $T_n(\tilde{u}) = \tilde{u}$ we have

$$u\left(\frac{1}{2n} + x\right) \; = \; \tilde{u}\left(-\frac{1}{2n} - x\right) \; = \; -\tilde{u}\left(-\frac{1}{2n} - x + \frac{1}{n}\right) \; = \; -u\left(\frac{1}{2n} - x\right), \tag{11}$$

for all $x \in (0, 1/(2n))$, i.e., the function u is odd with respect to $x = 1/(2n)$ on the subinterval $(0, 1/n)$. This in turn implies that the average of u vanishes over $(0, 1/n)$, and since the functions $\cos(k\pi x/L)$ for $L = 1/n$ and $k \in \mathbb{N}_0$ form a complete orthogonal set in $L^2((0, 1/n))$, one obtains that

$$u \in X_a \quad \text{ if and only if } \quad u(x) \; = \; \sum_{k \in n\mathbb{N}} a_k \cos(k \pi x) \; = \; \sum_{\ell \in \mathbb{N}} a_{\ell n} \cos(\ell n \pi x) \; ,$$

since the basis functions $\cos(\ell n \pi x)$ for $\ell \in \mathbb{N}$ clearly lie in X_a themselves. In other words, the function u is contained in X_a if and only if its cosine Fourier series contains only terms corresponding to wave numbers k which are multiples of n.

We now turn our attention to the spaces X_b and X_c. Due to Lemma 2.9, cosine Fourier expansions of a function u in either of these spaces can only contain terms for wave numbers which are not divisible by n. Furthermore, one can verify by direct inspection that

$$\begin{aligned} T_n^n \cos(k\pi x) &= \cos(k\pi x) && \text{if and only if} && n \equiv k \mod 2 \; , \\ T_n^n \cos(k\pi x) &= -\cos(k\pi x) && \text{if and only if} && n \not\equiv k \mod 2 \; . \end{aligned}$$

This gives the characterizations

$$\begin{aligned}
\text{Even } n\text{:}\quad & u \in X_b && \text{if and only if} && u(x) = \sum_{k \notin n\mathbb{N}\,,\; k \text{ even}} a_k \cos(k\pi x) \;, \\
& u \in X_c && \text{if and only if} && u(x) = \sum_{k \notin n\mathbb{N}\,,\; k \text{ odd}} a_k \cos(k\pi x) \;, \\
\text{Odd } n\text{:}\quad & u \in X_b && \text{if and only if} && u(x) = \sum_{k \notin n\mathbb{N}\,,\; k \text{ odd}} a_k \cos(k\pi x) \;, \\
& u \in X_c && \text{if and only if} && u(x) = \sum_{k \notin n\mathbb{N}\,,\; k \text{ even}} a_k \cos(k\pi x) \;.
\end{aligned}$$

We would like to point out that these characterizations immediately imply the statements of Table 1. Furthermore, the above characterizations remain valid without change for the image space decomposition $Y = Y_a \oplus Y_b \oplus Y_c$, as long as one considers the cosine Fourier series in a formal sense and its convergence in the norm defined in Y. For more details, we refer the reader to [24].

2.3. **Equivariance properties of the nonlinear operator.** We close this section by establishing the equivariance properties of the nonlinear diblock copolymer operator $F : X \to Y$ defined in (3) and (4), with a particular emphasis on how this operator and its Fréchet derivative L defined in (5) interacts with the space decompositions of X and Y from the last subsection. As mentioned in the introduction, throughout this paper we consider the one-dimensional domain $\Omega = (0, 1)$, the total mass $\mu = 0$, and the odd nonlinearity $f(u) = u - u^3$. We would like to point out, however, that the results of this section remain valid for any smooth odd function $f : \mathbb{R} \to \mathbb{R}$, and are in fact formulated for that case.

Throughout this section, we consider mapping properties of operators between the spaces X_τ and Y_τ introduced in the last section. In order to keep the notation as simple as possible, we will use the same letter for a function in X_τ and its extension in $H^2_{per}(\mathbb{R})$, and similarly for elements in Y_τ and their extensions in $H^{-2}_{per}(\mathbb{R})$. Thus, we can consider the diblock copolymer operator F both as an operator between X and Y, as well as an operator of the form $F : H^2_{per}(\mathbb{R}) \to H^{-2}_{per}(\mathbb{R})$.

It has already been stated several times that our main focus is the verification of a special kind of symmetry-breaking pitchfork bifurcation. Thus, the equilibrium solutions on the bifurcating branch will exhibit different, and in fact fewer, symmetry properties than the solutions on the primary steady state branch. This primary branch is characterized by invariance with respect to the symmetry operator T_n defined in (2), and the following first lemma shows that both F and its partial derivative $D_\lambda F$ respect this symmetry.

Lemma 2.12 (First equivariance properties)**.** *Let $\Omega = (0, 1)$, consider the total mass $\mu = 0$, and let f be a smooth and odd nonlinearity. Moreover, let $F : X \to Y$ be defined as in (3) and (4). Then for every $u \in X_a$ we have both $F(\lambda, u) \in Y_a$ and the inclusion $D_\lambda F(\lambda, u) \in Y_a$.*

Proof. Let $u \in X_a$ be arbitrary. Then its extension in $H^2_{per}(\mathbb{R})$ is even, and due to the properties of the Laplacian and the Nemitski operator f the same is true for both $F(\lambda, u)$ and $D_\lambda F(\lambda, u)$. Moreover, the oddness of f and $T_n u = u$ imply

$$T_n f(u(x)) = -f(u(x + 1/n)) = f(-u(x + 1/n)) = f(u(x)) \;.$$

In combination with $T_n\Delta = \Delta T_n$ one therefore obtains $T_n F(\lambda, u) = F(\lambda, u)$, and this in turn implies $F(\lambda, u) \in Y_a$, see also Definition 2.10. The remaining inclusion for $D_\lambda F(\lambda, u) = -\Delta f(u) - \sigma u$ can be verified analogously. □

In view of this lemma, one can study the equilibrium problem $F(\lambda, u) = 0$ restricted to the symmetry space X_a, and this will provide us with a primary solution branch in X_a. The next two auxiliary results address how first- and second-order partial derivatives of F interact with the various symmetry spaces. The resulting inclusions are central for our bifurcation analysis.

Lemma 2.13 (Equivariance properties of $D_u F$ and $D_{\lambda u} F$)**.** *Let $\Omega = (0, 1)$, consider the total mass $\mu = 0$, and let f be a smooth and odd nonlinearity. Furthermore, let $F : X \to Y$ be defined as in (3) and (4). Then for arbitrary $\lambda \in \mathbb{R}$ and $u \in X_a$, and every $\tau \in \{a, b, c\}$ we have the inclusions*

$$D_u F(\lambda, u)[X_\tau] \subset Y_\tau \qquad \text{and} \qquad D_{\lambda u} F(\lambda, u)[X_\tau] \subset Y_\tau \; . \tag{12}$$

In addition, we have

$$R(D_u F(\lambda, u)) = D_u F(\lambda, u)[X_a] \oplus D_u F(\lambda, u)[X_b] \oplus D_u F(\lambda, u)[X_c] \; , \tag{13}$$

as well as both

$$D_u F(\lambda, u)[X_\tau] = R(D_u F(\lambda, u)) \cap Y_\tau \qquad \text{and} \qquad P(Y_\tau) \subset Y_\tau \tag{14}$$

for all $\tau \in \{a, b, c\}$, where $P : Y \to Y$ denotes the orthogonal projection which was introduced in Definition 2.1.

Proof. Let $\lambda \in \mathbb{R}$ and $u \in X_a$ be fixed, and consider an arbitrary $v \in X$. For notational convenience in this proof, we introduce the abbreviation $L = D_u F(\lambda, u)$. Since we assumed that f is an odd function, its derivative f' is even. This immediately implies

$$\begin{aligned} T_n \left(f'(u(x)) v(x) \right) &= -f'(u(x + 1/n)) v(x + 1/n) \\ &= f'(-u(x))(-v(x + 1/n)) \; = \; f'(u(x)) T_n v(x) \; . \end{aligned}$$

Since the operator T_n also commutes with each of the other two terms in the explicit representation (5) of $D_u F(\lambda, u)[v]$, one therefore obtains $T_n L[v] = L[T_n v]$. This in turn implies that the inclusion $v \in X_\tau$ readily implies $D_u F(\lambda, u)[v] \in Y_\tau$, where we again refer the reader to Definition 2.10. The statement for the second-order partial derivative $D_{\lambda u} F(\lambda, u)[v] = -\Delta(f'(u)v) - \sigma v$ can be established completely analogously, and this completes the proof of (12).

The just-established (12) implies that the spaces $L[X_a]$, $L[X_b]$, and $L[X_c]$ form a direct sum, since Y_a, Y_b, and Y_c do. We also have $L[X_a] \oplus L[X_b] \oplus L[X_c] \subset R(L)$, and the reverse inclusion follows from $R(L) \ni y = Lx = Lx_a + Lx_b + Lx_c$, if we write $x = x_a + x_b + x_c \in X_a \oplus X_b \oplus X_c = X$. This implies (13).

As for (14), let $\tau \in \{a, b, c\}$. Then one obviously has $L[X_\tau] \subset R(L) \cap Y_\tau$. To verify the opposite inclusion, let $y \in R(L) \cap Y_\tau$ be arbitrary. Then $y = Lx$, and we can again decompose x in the form $x = x_a + x_b + x_c \in X_a \oplus X_b \oplus X_c$. But this in turn yields the inclusion $y = Lx_a + Lx_b + Lx_c \in Y_a \oplus Y_b \oplus Y_c$, and $y \in Y_\tau$ in combination with the properties of direct sums immediately imply $y = Lx_\tau \in L[X_\tau]$. In order to establish the inclusion statement regarding the orthogonal projection P, one just has to note that every $y \in Y$ can be written uniquely as $y = y_a + y_b + y_c \in Y_a \oplus Y_b \oplus Y_c$, and that for $\tau \in \{a, b, c\}$ one further has $y_\tau = y_{\tau,1} + y_{\tau,2}$, where $y_{\tau,1} \in L(X_\tau)$

and $y_{\tau,2}$ is contained in the orthogonal complement of $L(X_\tau)$ in Y_τ, which might in fact be trivial. Altogether, this gives the decomposition

$$y = y_{a,1} + y_{a,2} + y_{b,1} + y_{b,2} + y_{c,1} + y_{c,2}$$

into pairwise orthogonal elements, and one can easily see that $Py = y_{a,2} + y_{b,2} + y_{c,2}$. From this, the last statement follows readily, and the proof of the lemma is complete. □

Lemma 2.14 (Equivariance properties of $D_{uu}F$)**.** *Let $\Omega = (0,1)$, consider the total mass $\mu = 0$, and let f be a smooth odd nonlinearity. Furthermore, let $F : X \to Y$ be defined as in (3) and (4). Then for all $\lambda \in \mathbb{R}$ and $u \in X_a$ the following inclusions are satisfied:*

$$\begin{array}{llcl} (i) & D_{uu}F(\lambda,u)[X_a, X_a] & \subset & Y_a \\ (ii) & D_{uu}F(\lambda,u)[X_a \oplus X_b, X_a \oplus X_b] & \subset & Y_a \oplus Y_b \\ (iii) & D_{uu}F(\lambda,u)[X_a, X_b] & \subset & Y_b \\ (iv) & D_{uu}F(\lambda,u)[X_c, X_c] & \subset & Y_a \oplus Y_b \\ (v) & D_{uu}F(\lambda,u)[X_a \oplus X_b, X_c] & \subset & Y_c \end{array} \qquad (15)$$

Note that due to the symmetry of $D_{uu}F(\lambda,u)$, the order of the two arguments in (iii) and (v) does not matter.

Proof. Recall that we have $D_{uu}F(\lambda,u)[v,w] = -\Delta(\lambda f''(u)vw)$ for all $v,w \in X$, and that due to the oddness of f the second derivative f'' is also an odd function. Then for $u \in X_a$ one obtains the identity

$$\begin{aligned} T_n\left(f''(u(x))v(x)w(x)\right) &= -f''(u(x+1/n))v(x+1/n)w(x+1/n) \\ &= f''(-u(x+1/n))(-v(x+1/n))(-w(x+1/n)) \\ &= f''(u(x))T_nv(x)T_nw(x) \ , \end{aligned}$$

which in turn readily implies $T_nD_{uu}F(\lambda,u)[v,w] = D_{uu}F(\lambda,u)[T_nv,T_nw]$. With this formula at hand one can now establish the claims.

Note that if $v,w \in X_a$, then we have both $T_nv = v$ and $T_nw = w$, and the first statement follows. Similarly, if $v,w \in X_a \oplus X_b$, then $T_n^nv = v$ and $T_n^nw = w$, which yields the second statement. In addition, if we assume $v,w \in X_c$, then $T_n^nv = -v$ and $T_n^nw = -w$, and from this one can obtain *(iv)*.

We now turn our attention to *(iii)*. If $v \in X_a$ and $w \in X_b$, then for all $k \in \mathbb{N}$ one obtains $T_n^kD_{uu}F(\lambda,u)[v,w] = D_{uu}F(\lambda,u)[T_n^kv,T_n^kw] = D_{uu}F(\lambda,u)[v,T_n^kw]$. But this gives

$$\sum_{k=0}^{n-1} T_n^k D_{uu}F(\lambda,u)[v,w] = D_{uu}F(\lambda,u)\left[v, \sum_{k=0}^{n-1} T_n^k w\right] = 0 \ ,$$

which in turn implies $D_{uu}F(\lambda,u)[v,w] \in Y_b$. This establishes *(iii)*.

Finally, suppose that $v \in X_a \oplus X_b$ and $w \in X_c$. Then $T_n^nv = v$ and $T_n^nw = -w$, and therefore $T_n^nD_{uu}F(\lambda,u)[v,w] = D_{uu}F(\lambda,u)[T_n^nv,T_n^nw] = -D_{uu}F(\lambda,u)[v,w]$, which yields *(v)*. This completes the proof of the lemma. □

After these preparations we can now turn our attention to the main result of this section. It shows that under our Hypotheses 2.2 and 2.3 the kernel function at a potential bifurcation point has to be contained in one of the spaces X_a, X_b, and X_c. In addition, we obtain easily testable conditions that establish the precise

	$\phi_0 \in X_b$	$\phi_0 \in X_c$
n even	$\phi_0\left(\frac{1}{2n}\right) \neq 0, \quad \phi_0(0) + \phi_0(1) \neq 0$	$\phi_0(0) - \phi_0(1) \neq 0$
n odd	$\phi_0\left(\frac{1}{2n}\right) \neq 0, \quad \phi_0(0) - \phi_0(1) \neq 0$	$\phi_0(0) + \phi_0(1) \neq 0$

Table 2. Easily verifiable conditions which ensure whether the kernel function ϕ_0 is contained in X_b or in X_c. These conditions change depending on whether n in the definition of the symmetry operator T_n is even or odd. Since all of these conditions are simple inequality checks for specific function values, they can readily be rigorously validated using interval arithmetic.

space containing the kernel function. This will be essential for the bifurcation result and its rigorous verification via extended systems in the next section.

Proposition 2.15 (Kernel function properties). *Let $\Omega = (0,1)$, consider the total mass $\mu = 0$, and let the function f be a smooth and odd nonlinearity. Furthermore, let $F : X \to Y$ be defined as in (3) and (4), and suppose that Hypotheses 2.2 and 2.3 are satisfied, that is, we have both $F(\lambda_0, u_0) = 0$ and $L\phi_0 = D_u F(\lambda_0, u_0)[\phi_0] = 0$, and the null space of L is one-dimensional. In addition, we assume that the equilibrium u_0 is contained in the symmetry space X_a. Then the following statements hold.*

(a) The kernel function ϕ_0 is automatically contained in either X_a, or X_b, or X_c.
(b) By verifying one of the conditions in the left or right columns of Table 2, one can easily establish whether $\phi_0 \in X_b$ or $\phi_0 \in X_c$ is satisfied, respectively.
(c) If the kernel function satisfies $\phi_0 \notin X_a$, then the linearization $L : X_a \to Y_a$ is bijective. In particular, we have $L[X_a] = Y_a$ in this case.

Proof. We begin by verifying *(a)*. Since $\phi_0 \in X$, we can find $\phi_\tau \in X_\tau$ for $\tau = a, b, c$ such that the identity $\phi_0 = \phi_a + \phi_b + \phi_c$ is satisfied. But then (12) implies $L[\phi_\tau] \in Y_\tau$, i.e., we have $0 = L[\phi_0] = L[\phi_a] + L[\phi_b] + L[\phi_c] \in Y_a \oplus Y_b \oplus Y_c$, and the properties of direct sums therefore furnish $L[\phi_\tau] = 0$ for all $\tau = a, b, c$. Since $\phi_0 \neq 0$, we have to have $\phi_\tau \neq 0$ for at least one $\tau \in \{a, b, c\}$. Since the null space of L is one-dimensional, one then has to have the equality $\phi_0 = \alpha\phi_\tau \in X_\tau$ for some $\alpha \neq 0$, which establishes the claim.

Consider now the statement in *(b)* and suppose for the moment that the integer n in the definition of T_n is even. Then in view of Lemma 2.8, see also Table 1, every function in the direct sum $X_a \oplus X_b$ is even with respect to $x = 1/2$, while every function in X_c is odd with respect to the center of the interval Ω. By taking the contrapositive, if we know that the inequality $\phi_0(0) + \phi_0(1) \neq 0$ is true, then ϕ_0 cannot be odd with respect to $x = 1/2$, and therefore has to be in X_a or X_b. Similarly, the inequality $\phi_0(0) - \phi_0(1) \neq 0$ shows that ϕ_0 is not even, and so it has to be in X_c. The case of odd n can be treated similarly.

It remains to show that if we have the inclusion $\phi_0 \in X_a \cup X_b$, as well as $\phi_0(1/(2n)) \neq 0$, then necessarily one has $\phi_0 \in X_b$. This, however, follows immediately from (11), where it was demonstrated that every function in X_a is odd with respect to $x = 1/(2n)$, and therefore has to vanish at $1/(2n)$. This completes the proof of *(b)*.

Finally, we turn our attention to the statement in *(c)*. Suppose therefore that ϕ_0 is contained in either X_b or X_c. Since we assumed $T_n u_0 = u_0$, the function u_0 satisfies homogeneous Neumann boundary conditions at both $x = 0$ and at $x = 1/n$. Now let u_s denote the restriction of u_0 to the interval $\Omega_s = (0, 1/n)$. Then we still have $F(\lambda_0, u_s) = 0$ on this interval, with the same Neumann boundary conditions that were considered for the original equation on $\Omega = (0, 1)$. Define the function spaces X_s and Y_s as the spaces corresponding to X and Y, but consisting of functions restricted to the smaller interval Ω_s, which in the case of X_s satisfy homogeneous Neumann boundary conditions on $\partial\Omega_s$. Finally, define $L_s = D_u F(\lambda_0, u_s)|_{X_s}$, i.e., via restriction to the interval $(0, 1/n)$.

We begin by verifying $L[X_s] = Y_s$. For this, assume that $Y_s \setminus L_s[X_s] \neq \emptyset$. Then there exists a nontrivial element which is not in the range of L_s. Therefore, since also the restricted operator is Fredholm with index 0 according to [13, Proposition 2.15], there has to be a nontrivial null space element $\phi_s \in N(L_s)$. Recall that any function in X_a is uniquely defined by its values on $\Omega_s = (0, 1/n)$. Thus, since ϕ_s satisfies homogeneous Neumann boundary conditions on Ω_s, there is a corresponding unique function $\phi_a \in X_a$ defined on $\Omega = (0, 1)$ and such that $\phi_a = \phi_s$ on $(0, 1/n)$. Moreover, the fact that $L_s\phi_s = 0$ on Ω_s immediately implies that $L\phi_a = 0$ on Ω. Therefore, the function $\phi_a \in X_a$ is contained in the null space $N(L)$. However, this is a contradiction, since we have assumed that $N(L)$ is one-dimensional, and that it is spanned by a function ϕ_0 which is contained in either X_b or X_c. Thus we conclude that $L[X_a] = Y_a$. Since the above argument also directly implies $N(L_s) = \{0\}$, this completes the proof of the proposition. □

3. **Cyclic equivariant pitchfork bifurcations.**

After the preparations of the last section, we now show that the cyclic action of the symmetry operator T_n, through its derived space decomposition $X = X_a \oplus X_b \oplus X_c$, does indeed give rise to symmetry-breaking bifurcations. More precisely, we consider the scenarios indicated in Table 3. In this table, we distinguish between the parity of the integer n and the symmetry with respect to $x = 1/2$ of the kernel function ϕ_0. This leads to four different bifurcation scenarios, all of which break the $\mathbb{Z}_{2n}$-symmetry of the equilibrium solution $u_0 \in X_a$, based on whether one has $\phi_0 \in X_b$ or $\phi_0 \in X_c$. Recall that the latter two conditions can easily be verified using the tests listed in Table 2. While all of these scenarios are covered by the theory developed in the present paper, cases (b) and (c) could already be established using the results of [13]. However, the cases (a) and (d) are new and do require our new approach, and they cover all of the situations shown in Figures 1 and 2.

To develop this new approach, we proceed as follows. In Section 3.1 we use a standard Lyapunov-Schmidt reduction based on our underlying space decomposition to provide a sufficient condition for the existence of a $\mathbb{Z}_{2n}$-symmetry breaking pitchfork bifurcation. After that, Section 3.2 demonstrates that the assumptions of this result can be verified using the existence of an isolated zero of a suitable extended system. This reformulation of the existence result makes it amenable to verification via computer-assisted proof techniques.

3.1. **A sufficient condition for pitchfork bifurcation.**

We begin by concentrating on the derivation of a sufficient condition for the existence of a pitchfork bifurcation which breaks the $\mathbb{Z}_{2n}$-symmetry. For this, we rely on the Lyapunov-Schmidt reduction result in Proposition 3.1 below, which gives a general method

	ϕ_0 even	ϕ_0 odd
n even	(a) u_0 even $\phi_0 \in X_b$	(b) u_0 even $\phi_0 \in X_c$
n odd	(c) u_0 odd $\phi_0 \in X_c$	(d) u_0 odd $\phi_0 \in X_b$

TABLE 3. Symmetry-breaking pitchfork bifurcation scenarios if the equilibrium solution u_0 satisfies $u_0 \in X_a$, i.e., we have $T_n u_0 = u_0$. Throughout the table, the labels even and odd refer to symmetries with respect to the center point $x = 1/2$ of the domain $\Omega = (0, 1)$. Notice that (b) and (c) are covered by our previous $\mathbb{Z}_2$-pitchfork bifurcation theorem [13]. In contrast, the remaining two cases (a) and (d) correspond to the new scenarios depicted in Figures 1 and 2, respectively.

of reducing the bifurcation problem from an infinite-dimensional problem to a bifurcation problem on a one-dimensional subspace. Our formulation is completely analogous to the one used in [13]. Thus, we refer the reader to this paper for the full proof and merely provide a brief sketch below to keep the current paper self-contained.

Proposition 3.1 (Lyapunov-Schmidt reduction)**.** *Let $\Omega = (0, 1)$, consider the total mass $\mu = 0$, and let the function f be a smooth and odd nonlinearity. Furthermore, let $F : X \to Y$ be defined as in (3) and (4), and suppose that Hypotheses 2.2 and 2.3 are satisfied, i.e., we have both $F(\lambda_0, u_0) = 0$ and $L\phi_0 = D_u F(\lambda_0, u_0)[\phi_0] = 0$, and the null space of L is one-dimensional. Finally, let P and Q denote the orthogonal projections from Definition 2.1.*

Then there exist a neighborhood Λ_0 of λ_0, a neighborhood V_0 of $v_0 = Qu_0 \in N(L)$, a smooth function $W : \Lambda_0 \times V_0 \to \tilde{X}$, as well as a smooth real-valued function b which is defined in a neighborhood of the point $(\lambda_0, 0) \in \mathbb{R}^2$ such that the following hold:

(a) If (λ, α) is sufficiently close to the point $(\lambda_0, 0) \in \mathbb{R}^2$ and satisfies $b(\lambda, \alpha) = 0$, then we have

$$F(\lambda, u) = 0 \quad \text{for} \quad u = v_0 + \alpha\phi_0 + W(\lambda, v_0 + \alpha\phi_0) \ .$$

(b) Conversely, if (λ, u) is close enough to (λ_0, u_0) and solves $F(\lambda, u) = 0$, then for α defined via $v_0 + \alpha\phi_0 = Qu$ we have $b(\lambda, \alpha) = 0$ and $u = Qu + W(\lambda, Qu)$.

In other words, the solution set of $b(\lambda, \alpha) = 0$ in a neighborhood of $(\lambda_0, 0) \in \mathbb{R}^2$ is in one-to-one correspondence with the solution set of $F(\lambda, u) = 0$ in a neighborhood of (λ_0, u_0).

Proof. Due to the properties of the projections P and Q, if we associate with every element $u \in X$ the two elements $v = Qu$ and $w = (I - Q)u$, then we clearly have the identity $u = v + w \in N(L) \oplus \tilde{X}$. Moreover, the nonlinear problem $F(\lambda, u) = 0$ is equivalent to the system

$$PF(\lambda, v + w) = 0 \qquad \text{and} \qquad G(\lambda, v, w) := (I - P)F(\lambda, v + w) = 0 \ . \qquad (16)$$

One can show that the function $G : \mathbb{R} \times N(L) \times \tilde{X} \to R(L)$ introduced in the second equation has an invertible Fréchet derivative $D_w G(\lambda_0, v_0, u_0 - v_0) \in \mathcal{L}(\tilde{X}, R(L))$, and therefore the implicit function theorem implies that given (λ, v) near (λ_0, v_0), there exists a unique $w = W(\lambda, v)$ in $\tilde{X}$ such that $G(\lambda, v, W(\lambda, v)) = 0$. Since $N(L)$ is one-dimensional, each $v \in N(L)$ has a unique representation of the form $v_0 + \alpha\phi_0$. Given a nontrivial $\psi_0^* \in Y^*$ such that $R(L) = N(\psi_0^*)$, the function b is then defined as

$$b(\lambda, \alpha) = \psi_0^*(PF(\lambda, v_0 + \alpha\phi_0 + W(\lambda, v_0 + \alpha\phi_0))) . \tag{17}$$

The statements of the proposition now follow easily from the fact that the solutions of the original problem $F(\lambda, u) = 0$ are in one-to-one correspondence with the solutions of (16). □

The above proposition is the standard version of the Lyapunov-Schmidt reduction, which does not account for any symmetry properties of the nonlinear operator F. Note, however, that we do require that the two projections P and Q are orthogonal projections, and in that way the results from the last section allow us to make a number of additional deductions as long as we assume that the equilibrium solution u_0 is contained in X_a, while the kernel function ϕ_0 is contained in either X_b or X_c. More precisely, we will soon see that the following hold:

- Since the spaces in the decomposition $X = X_a \oplus X_b \oplus X_c$ are pairwise orthogonal, we automatically obtain $Q[X_a] = \{0\}$.
- In view of $F(\lambda, X_a) \subset Y_a$ and the assumption on ϕ_0, there exists a unique branch of equilibrium solutions through (λ_0, u_0) which is contained in X_a, and the application of the projection Q transforms this branch into a trivial solution branch in $\mathbb{R} \times N(L)$.
- The construction of the bifurcation equation $b(\lambda, \alpha) = 0$ in the above proposition then readily implies that $b(\lambda, 0) = 0$ for all λ in a neighborhood of λ_0. In fact, since we also assumed the oddness of the nonlinearity f, we can even make statements about the vanishing of certain derivatives of the function b.

These observations are explained in more detail in the following main result of this subsection. It provides conditions under which the $\mathbb{Z}_{2n}$-equivariance of the last section forces a symmetry-breaking pitchfork bifurcation. This result is similar in spirit to [13, Proposition 2.11], as well as to the classical result [6].

Theorem 3.2 (Existence of $\mathbb{Z}_{2n}$-symmetry breaking pitchfork bifurcation)**.** *Let $\Omega = (0, 1)$, consider the mass $\mu = 0$, and let the function f be a smooth and odd nonlinearity. Furthermore, let $F : X \to Y$ be defined as in (3) and (4), and suppose that Hypotheses 2.2 and 2.3 are satisfied, i.e., we have both $F(\lambda_0, u_0) = 0$ and $L\phi_0 = D_uF(\lambda_0, u_0)[\phi_0] = 0$, and the null space of L is one-dimensional. Finally, let P and Q denote the orthogonal projections from Definition 2.1. In addition, suppose that $u_0 \in X_a$ and that $\phi_0 \in X_\tau$ for $\tau \in \{b, c\}$. Then there exists a unique function $\xi_0 \in X_a$ such that*

$$L\xi_0 + (I - P)D_\lambda F(\lambda_0, u_0) = 0 , \tag{18}$$

and if we further suppose that the nondegeneracy condition

$$D_{\lambda u}F(\lambda_0, u_0)[\phi_0] + D_{uu}F(\lambda_0, u_0)[\phi_0, \xi_0] \notin R(L) \tag{19}$$

is satisfied, then the point (λ_0, u_0) is a pitchfork bifurcation point for the nonlinear operator F.

Proof. Consider the function $G(\lambda, v, w)$ defined in (16), as well as $b(\lambda, \alpha)$ introduced in (17). Furthermore, let $v_0 = Qu_0 \in N(L)$ as in Proposition 3.1. Then $u_0 \in X_a$, combined with the fact that $\phi_0 \in X_\tau$ for $\tau \neq a$, that $Q : X \to N(L)$ is an orthogonal projection, and the orthogonality statement from Lemma 2.9, immediately yield both $v_0 = 0$ and $N(L|_{X_a}) = \{0\}$. Also, the definition of ψ_0^* and [13, Table 1] yield

$$b_\alpha(\lambda_0, 0) = \psi_0^* D_u F(\lambda_0, u_0) = 0 \; .$$

The oddness of the nonlinearity f and $\mu = 0$ further show that $F(\lambda, -u) = -F(\lambda, u)$ for arbitrary $\lambda \in \mathbb{R}$ and $u \in X$, and this in turn implies

$$G(\lambda, -v, -w) = (I - P)F(\lambda, -(v + w)) = -(I - P)F(\lambda, v + w) = -G(\lambda, v, w) \; .$$

Thus, the function $w = -W(\lambda, v) \in \tilde{X}$ solves the equation $G(\lambda, -v, w) = 0$, and the uniqueness property of W established in Proposition 3.1 then gives $-W(\lambda, v) = W(\lambda, -v)$. One then obtains

$$\begin{aligned} -b(\lambda, \alpha) &= -\psi_0^* PF(\lambda, \alpha\phi_0 + W(\lambda, \alpha\phi_0)) \\ &= \psi_0^* PF(\lambda, -\alpha\phi_0 + W(\lambda, -\alpha\phi_0)) = b(\lambda, -\alpha) \; , \end{aligned}$$

which implies the trivial solution $b(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$, as well as $b_{\alpha\alpha}(\lambda, 0) = 0$.

If we now again apply Proposition 3.1, then the trivial solution of b gives rise to the smooth solution curve $\lambda \mapsto W(\lambda, 0) \in \tilde{X}$. In fact, one can show that this solution branch lies in X_a, since in view of $N(L|_{X_a}) = \{0\}$ we can apply the implicit function theorem to the restriction $F : \mathbb{R} \times X_a \to Y_a$.

In order to establish the bifurcating branch which breaks the X_a-symmetry, we define a function r in a neighborhood of $(\lambda_0, 0)$ by setting

$$r(\lambda, \alpha) = \begin{cases} \dfrac{b(\lambda, \alpha)}{\alpha} & \text{for} \quad \alpha \neq 0 \; , \\ b_\alpha(\lambda, 0) & \text{for} \quad \alpha = 0 \; . \end{cases}$$

One can easily show that r is smooth. Moreover, one can show as in [13, Proposition 2.11] that r has the expansion

$$r(\lambda_0 + \nu, \alpha) = \nu b_{\lambda\alpha}(\lambda_0, 0) + \frac{\nu^2}{2} b_{\lambda\lambda\alpha}(\lambda_0, 0) + \frac{\alpha\nu}{2} b_{\lambda\alpha\alpha}(\lambda_0, 0) + \frac{\alpha^2}{6} b_{\alpha\alpha\alpha}(\lambda_0, 0) + R(\nu, \alpha)$$

with $R(\nu, \alpha) = O(\|(\nu, \alpha)\|^3)$. One clearly has $r(\lambda_0, 0) = b_\alpha(\lambda_0, 0) = 0$. Furthermore, it was shown in [13, Proposition 2.11] that

$$r_\lambda(\lambda_0, 0) = b_{\lambda\alpha}(\lambda_0, 0) = \psi_0^* D_{\lambda u} F(\lambda_0, u_0)[\phi_0] + \psi_0^* D_{uu} F(\lambda_0, u_0)[\phi_0, \xi_0] \neq 0 \; ,$$

with ξ_0 as defined uniquely in (18). The implicit function theorem then yields a smooth function $\alpha \mapsto h(\alpha)$ which is defined near $\alpha = 0$, satisfies $h(0) = \lambda_0$, and such that in a neighborhood of $(0, 0)$ one has

$$r(\lambda, \alpha) = 0 \quad \text{if and only if} \quad \lambda = h(\alpha) \; .$$

This establishes the second solution branch $\alpha \mapsto \alpha\phi_0 + W(h(\alpha), \alpha\phi_0)$. Finally, one can follow the proof of [13, Proposition 2.11] verbatim to show that the two solution curves together form an actual pitchfork bifurcation. This is accomplished by deriving an explicit formula for $h'(0)$ and showing that it vanishes, which then completes the proof of the theorem. □

We would like to point out that in the above theorem, we classify a bifurcation as a pitchfork bifurcation if the bifurcating solution branch is tangential to the space $\{\lambda_0\} \times X$. In order to actually get the parabolic shape of the bifurcating branch that is usually associated with the pitchfork bifurcation, one needs to verify another

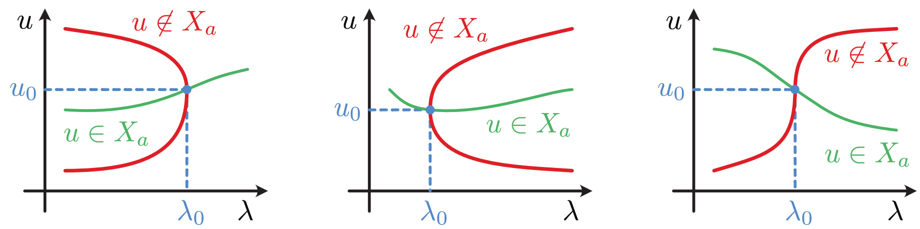

FIGURE 4. Possible symmetry-breaking pitchfork bifurcation scenarios in Theorem 3.2. In the generic case, i.e., if the constant ρ defined in (20) is nonzero, one observes a classic pitchfork bifurcation — and the cases $\rho > 0$ and $\rho < 0$ are shown in the first and second panels, respectively. However, in the case $\rho = 0$ one could observe a situation depicted in the right-most panel.

non-degeneracy condition. To illustrate this, one can show that the function h constructed in the above proof satisfies not only $h(0) = \lambda_0$ and $h'(0) = 0$, but also the identity $h''(0) = -\rho/3$, where the constant ρ is given by

$$\rho = \frac{\psi_0^* D_{uuu} F(\lambda_0, u_0)[\phi_0, \phi_0, \phi_0] + 3\psi_0^* D_{uu} F(\lambda_0, u_0)[\phi_0, \zeta_0]}{\psi_0^* D_{\lambda u} F(\lambda_0, u_0)[\phi_0] + \psi_0^* D_{uu} F(\lambda_0, u_0)[\phi_0, \xi_0]} , \tag{20}$$

and $\zeta_0 \in X_a$ is defined by the equation

$$D_u F(\lambda_0, u_0)[\zeta_0] + (I - P) D_{uu} F(\lambda_0, u_0)[\phi_0, \phi_0] = 0 .$$

If the ratio ρ is positive, then the solutions on the parabolic branch exist for $\lambda < \lambda_0$ close to the bifurcation point, if ρ is negative then they exist for $\lambda > \lambda_0$. If, on the other hand, one has $\rho = 0$, either half of the branch could lie on either side of λ_0. These cases are illustrated in Figure 4. For more details, we refer the reader to the discussion in [13].

3.2. **Pitchfork bifurcations via extended systems.** With Theorem 3.2 we have established an explicit sufficient condition for the existence of a pitchfork bifurcation in the diblock copolymer model which is induced by the action of the $\mathbb{Z}_{2n}$-symmetry given by the operator $T_n u(x) = -u(x + 1/n)$. As was pointed out in [13], however, this condition is ill-suited if one would like to derive computer-assisted proofs for the existence of curves of such bifurcation points in a two-parameter setting. More useful in this situation is a reformulation of the existence result in terms of a zero finding problem for an extended system — and this reformulation can be adapted to our current setting.

To explain this in more detail, consider again the space decompositions $X = X_a \oplus X_b \oplus X_c$ and $Y = Y_a \oplus Y_b \oplus Y_c$ which were introduced in Definitions 2.7 and 2.10. In addition, consider a fixed element $\ell \in X^*$ in the dual space of X. We then introduce the following extended system for F, which is modeled after the one we used in [13]:

$$\begin{aligned} &\text{Solve} \quad \mathcal{F}_e(\lambda, u, v) = (0, 0, 0) \\ &\text{for} \quad \mathcal{F}_e : \left\{ \begin{array}{c} \mathbb{R} \times X_a \times X \to \mathbb{R} \times Y_a \times Y \\ (\lambda, u, v) \mapsto (\ell(v) - 1, F(\lambda, u), D_u F(\lambda, u)[v]) \end{array} \right. \end{aligned} \tag{21}$$

The operator $\mathcal{F}_e$ is well-defined in view of Lemma 2.12. Furthermore, its Fréchet derivative is an operator in $\mathcal{L}(\mathbb{R} \times X_a \times X,\ \mathbb{R} \times Y_a \times Y)$ which is explicitly given by

$$D\mathcal{F}_e(\lambda, u, v)[\tilde{\lambda}, \tilde{u}, \tilde{v}] = \Big(\ell(\tilde{v}) ,\ \ \tilde{\lambda} D_\lambda F(\lambda, u) + D_u F(\lambda, u)[\tilde{u}] , \quad (22)$$
$$\tilde{\lambda} D_{\lambda u} F(\lambda, u)[v] + D_{uu} F(\lambda, u)[v, \tilde{u}] + D_u F(\lambda, u)[\tilde{v}]\Big) .$$

As the next main result of this section shows, the existence of a nondegenerate zero of this extended system is equivalent to the sufficient condition for an $\mathbb{Z}_{2n}$-induced symmetry-breaking pitchfork bifurcation given in Theorem 3.2.

Theorem 3.3 ($\mathbb{Z}_{2n}$-symmetry breaking pitchfork bifurcation via extended systems)**.** *As before, consider the domain $\Omega = (0, 1)$, the total mass $\mu = 0$, and let the function f be a smooth and odd nonlinearity. Furthermore, let $F : X \to Y$ be defined as in (3) and (4). Then the following two statements hold.*

(a) Suppose that all assumptions of Theorem 3.2 are satisfied, and let $\ell \in X^*$ be such that $\ell(\phi_0) = 1$. Then the Fréchet derivative $D\mathcal{F}_e(\lambda_0, u_0, \phi_0)$ of the mapping in (21) is invertible, i.e., the solution $(\lambda_0, u_0, \phi_0) \in \mathbb{R} \times X_a \times X_\tau$ of the extended system

$$\mathcal{F}_e(\lambda, u, \phi) = (0, 0, 0) \quad (23)$$

is an isolated non-degenerate zero.

(b) Conversely, if there exists an $\ell \in X^*$ and a $\phi_0 \in X_b \cup X_c$ such that (λ_0, u_0, ϕ_0) is a zero of the map $\mathcal{F}_e$, and if the Fréchet derivative $D\mathcal{F}_e(\lambda_0, u_0, \phi_0)$ is invertible, then the nonlinear operator F satisfies all assumptions of Theorem 3.2.

In other words, the diblock copolymer equilibrium problem defined earlier in (1) undergoes a $\mathbb{Z}_{2n}$-symmetry breaking pitchfork bifurcation at the point (λ_0, u_0) in the sense of Theorem 3.2, if and only if $(\lambda_0, u_0, \phi_0) \in \mathbb{R} \times X_a \times X_\tau$ is a non-degenerate zero of (23) for $\tau \in \{b, c\}$. Note, however, that for this we consider $\mathcal{F}_e$ as an operator defined on $\mathbb{R} \times X_a \times X$, even though ϕ_0 has to be contained in X_τ.

Proof. We begin by establishing the validity of *(a)*. It is clear that the assumptions of Theorem 3.2, in combination with $\ell(\phi_0) = 1$, imply that $(\lambda_0, u_0, \phi_0) \in \mathbb{R} \times X_a \times X_\tau$ is a solution of (23), where $\tau \in \{b, c\}$.

In order to verify that the Fréchet derivative $D\mathcal{F}_e(\lambda_0, u_0, \phi_0)$ is one-to-one, suppose there exists $(\tilde{\lambda}, \tilde{u}, \tilde{v}) \in \mathbb{R} \times X_a \times X$ such that

$$D\mathcal{F}_e(\lambda_0, u_0, \phi_0)[\tilde{\lambda}, \tilde{u}, \tilde{v}] = (0, 0, 0) . \quad (24)$$

We show that this implies $(\tilde{\lambda}, \tilde{u}, \tilde{v}) = (0, 0, 0)$. Assume first that the inequality $\tilde{\lambda} \neq 0$ holds. Then in view of $\phi_0 \notin X_a$ and Proposition 2.15*(c)* we know that $L(X_a) = Y_a$. Lemma 2.12 and the definition of P imply $D_\lambda F(\lambda_0, u_0) \in Y_a = L(X_a) \subset R(L) \subset N(P)$, which in turn gives the identity $D_\lambda F(\lambda_0, u_0) = (I - P) D_\lambda F(\lambda_0, u_0)$. Now the second component of (24), which can be made explicit via (22), can be rewritten in the form

$$(I - P) D_\lambda F(\lambda_0, u_0) + L[\tilde{u}/\tilde{\lambda}] = 0 ,$$

and since $\xi_0 \in X_a$ is the unique solution of (18), this immediately furnishes $\xi_0 = \tilde{u}/\tilde{\lambda}$. The third component of (24) and (22) then gives

$$D_{\lambda u} F(\lambda_0, u_0)[\phi_0] + D_{uu} F(\lambda_0, u_0)[\phi_0, \xi_0] = -D_u F(\lambda_0, u_0)[\tilde{v}/\tilde{\lambda}] = -L[\tilde{v}/\tilde{\lambda}] \in R(L),$$

which contradicts our assumption (19), and we therefore obtain $\tilde{\lambda} = 0$. The second component of (24) and (22) implies $L[\tilde{u}] = 0$, and thus also $\tilde{u} \in N(L) \cap X_a = \{0\}$, i.e., we necessarily have $\tilde{u} = 0$. Substituting both $\tilde{\lambda} = 0$ and $\tilde{u} = 0$ into the third component finally gives $L\tilde{v} = 0$, as well as $\tilde{v} = \alpha\phi_0$. Together with the first component of (24) this further yields $0 = \ell(\alpha\phi_0) = \alpha\ell(\phi_0) = \alpha$, i.e., one has the identity $\tilde{v} = 0$. This completes the proof that $D\mathcal{F}_e(\lambda_0, u_0, \phi_0)$ is one-to-one.

We now show that $D\mathcal{F}_e(\lambda_0, u_0, \phi_0)$ is onto. For this, let $(\tau, y, z) \in \mathbb{R} \times Y_a \times Y$ be arbitrary, but fixed. We need to explicitly construct an inverse image under the Fréchet derivative. To this end, notice first that since $y \in Y_a = L(X_a)$ and $N(L) \cap X_a = \{0\}$, there exists a unique element $\tilde{u} \in X_a$ such that $L[\tilde{u}] = y$. In view of Hypothesis 2.3, the linear functional $\psi_0^* \in Y^*$ satisfies $R(L) = N(\psi_0^*)$. Now define

$$\tilde{\lambda} = \frac{\psi_0^*\left(z - D_{uu}F(\lambda_0, u_0)[\phi_0, \tilde{u}]\right)}{\psi_0^*\left(D_{\lambda u}F(\lambda_0, u_0)[\phi_0] + D_{uu}F(\lambda_0, u_0)[\phi_0, \xi_0]\right)} ,$$

where the denominator of this ratio is nonzero due to (19). A simple algebraic reformulation of this definition then leads to

$$\psi_0^*\left(\tilde{\lambda} D_{\lambda u}F(\lambda_0, u_0)[\phi_0] + D_{uu}F(\lambda_0, u_0)[\phi_0, \tilde{u} + \tilde{\lambda}\xi_0] - z\right) = 0 .$$

The choice of ψ_0^* shows that the argument in the above equation has to be contained in $R(L)$, and since $L[\beta\phi_0] = 0$ for any scalar β, there exists a $\tilde{v} \in X$ such that for every $\beta \in \mathbb{R}$ the equation

$$\tilde{\lambda} D_{\lambda u}F(\lambda_0, u_0)[\phi_0] + D_{uu}F(\lambda_0, u_0)[\phi_0, \tilde{u} + \tilde{\lambda}\xi_0] + L[\tilde{v} + \beta\phi_0] = z \qquad (25)$$

is satisfied. Now (18), together with $N(P) = R(L)$ and $D_\lambda F(\lambda_0, u_0) \in Y_a \subset R(L)$ give

$$\tilde{\lambda} D_\lambda F(\lambda_0, u_0) + L[\tilde{u} + \tilde{\lambda}\xi_0] = \tilde{\lambda}\left((I - P)D_\lambda F(\lambda_0, u_0) + L[\xi_0]\right) + L[\tilde{u}] = y. \qquad (26)$$

Finally, notice that $\xi_0 \in X_a$ yields $\tilde{u} + \tilde{\lambda}\xi_0 \in X_a$, and this in turn implies that for all $\beta \in \mathbb{R}$ the identities in (25) and (26) establish the second and third components of the desired equation

$$D\mathcal{F}_e(\lambda_0, u_0, \phi_0)[\tilde{\lambda},\ \tilde{u} + \tilde{\lambda}\xi_0,\ \tilde{v} + \beta\phi_0] = (\tau, y, z) .$$

It remains to choose β in such a way that the first component of the equation holds as well. Since $\ell(\phi_0) = 1$ by our earlier normalization, we need to solve the equation $\tau = \ell(\tilde{v} + \beta\phi_0) = \ell(\tilde{v}) + \beta$, which is clearly satisfied in we let $\beta = \tau - \ell(\tilde{v})$. This shows that $D\mathcal{F}_e(\lambda_0, u_0, \phi_0)$ is onto, and completes the proof of *(a)*.

We now turn our attention to the verification of *(b)*, i.e., we assume that there exists an $\ell \in X^*$ and a $\phi_0 \in X_b \cup X_c$ such that (λ_0, u_0, ϕ_0) is a zero of the map $\mathcal{F}_e$, and that the Fréchet derivative $D\mathcal{F}_e(\lambda_0, u_0, \phi_0) \in \mathcal{L}(\mathbb{R} \times X_a \times X,\ \mathbb{R} \times Y_a \times Y)$ is invertible. We need to show that all the assumptions of Theorem 3.2 are satisfied.

We begin by establishing Hypotheses 2.2 and 2.3. In view of $\mathcal{F}_e(\lambda_0, u_0, \phi_0) = 0$ and (21), one obtains $F(\lambda_0, u_0) = 0$, as well as $L[\phi_0] = D_uF(\lambda_0, u_0)[\phi_0] = 0$. Furthermore, due to $\ell(\phi_0) = 1$ we have $\dim N(L) \geq 1$. On the other hand, since the Fréchet derivative $D\mathcal{F}_e(\lambda_0, u_0, \phi_0)$ is invertible, it can be shown as in [13, Proof of Theorem 2.12] that we have in fact $\dim N(L) = 1$. Since in this paper it was also shown that L is a Fredholm operator with index zero, this establishes both Hypotheses 2.2 and 2.3. In particular, it follows that P and Q as defined in Definition 2.1 have rank one. Notice also that all the assumptions of Proposition 2.15 have been established, and *(c)* of this result, in combination with $\phi_0 \in X_b \cup X_c$, then immediately implies $L(X_a) = Y_a$.

In order to establish the remaining assumptions of Theorem 3.2, we first show that the equation in (18) has a unique solution $\xi_0 \in X_a$. We know that $u_0 \in X_a$, and this yields the inclusion $D_\lambda F(\lambda_0, u_0) \in Y_a \subset R(L)$. Definition 2.1 implies the equality $N(P) = R(L)$, and therefore

$$(I - P)D_\lambda F(\lambda_0, u_0) = D_\lambda F(\lambda_0, u_0) \in Y_a$$

holds. This implies the existence of a $\xi_0 \in X_a$ with $L\xi_0 = -(I - P)D_\lambda F(\lambda_0, u_0)$ in view of $L(X_a) = Y_a$, i.e., equation (18) holds. Moreover, this solution ξ_0 is uniquely determined, because if $\hat{\xi}_0 \in X_a$ were another solution, then

$$L\hat{\xi}_0 = -(I - P)D_\lambda F(\lambda_0, u_0) = L\xi_0,$$

i.e., one has $L(\hat{\xi}_0 - \xi_0) = 0$. But this yields the inclusion $\hat{\xi}_0 - \xi_0 \in N(L) \cap X_a = \{0\}$.

To conclude our proof, we only have to establish (19). For this, let z be any element of the complement $Y \setminus R(L)$. Due to the assumptions of *(b)* there exists a triple $(\tilde{\lambda}, \tilde{u}, \tilde{v}) \in \mathbb{R} \times X_a \times X$ such that the identity $D\mathcal{F}_e(\lambda_0, u_0, \phi_0)[\tilde{\lambda}, \tilde{u}, \tilde{v}] = (0, 0, z)$ is satisfied. Let ξ_0 again denote the unique solution to (18) from above. We assume first that $\tilde{\lambda} = 0$. Then the explicit form of the second component given in (22) yields the identity $0 = \tilde{\lambda} D_\lambda F(\lambda_0, u_0) + L\tilde{u} = L\tilde{u}$, which in turn furnishes the inclusion $\tilde{u} \in N(L) \cap X_a = \{0\}$, and thus $\tilde{u} = 0$. Therefore, another application of (22) implies

$$z = \tilde{\lambda} D_{\lambda u} F(\lambda_0, u_0)[\phi_0] + D_{uu} F(\lambda_0, u_0)[\phi_0, \tilde{u}] + L\tilde{v} = L\tilde{v} \ ,$$

which contradicts $z \notin R(L)$. Thus, our assumption concerning $\tilde{\lambda}$ was wrong, and we have to have the inequality $\tilde{\lambda} \neq 0$. But then (22) gives rise to $0 = \tilde{\lambda} D_\lambda F(\lambda_0, u_0) + L\tilde{u}$, and after division by $\tilde{\lambda}$ one obtains

$$D_\lambda F(\lambda_0, u_0) + L[\tilde{u}/\tilde{\lambda}] = 0 \ . \tag{27}$$

Notice that we already established earlier that $Y_a = L(X_a) \subset R(L) = N(P)$, as well as $D_\lambda F(\lambda_0, u_0) \in Y_a \subset N(P)$, and this immediately gives $PD_\lambda F(\lambda_0, u_0) = 0$. In combination with (27) one then obtains

$$(I - P)D_\lambda F(\lambda_0, u_0) + L[\tilde{u}/\tilde{\lambda}] = 0 \ ,$$

and since ξ_0 is the unique solution to this latter equation, we have to have $\xi_0 = \tilde{u}/\tilde{\lambda}$. Substituting this into the third component of $D\mathcal{F}_e(\lambda_0, u_0, \phi_0)[\tilde{\lambda}, \tilde{u}, \tilde{v}] = (0, 0, z)$, one finally obtains after a few algebraic reformulations

$$D_{\lambda u} F(\lambda_0, u_0)[\phi_0] + D_{uu} F(\lambda_0, u_0)[\phi_0, \xi_0] = z/\tilde{\lambda} - L\tilde{v} \ .$$

Clearly we have $L\tilde{v} \in R(L)$. Thus, if the right-hand side of this equation were contained in $R(L)$, then one would have to have $z \in R(L)$, which contradicts our original assumption. Therefore, the right-hand side cannot be an element of $R(L)$, and this establishes (19). This completes the proof of the theorem. □

The above result is remarkable in the sense that even though we are considering a completely different symmetry from the simple ones discussed in [13], we still obtain essentially the same sufficient existence condition for a symmetry-breaking pitchfork bifurcation via the extended system (21). All that changes is the restriction of the second argument u to the new symmetry space. In fact, a closer inspection of our results shows that the same approach should work in other situations as well, as long as a few basic assumptions are satisfied. These are collected in the following remark.

Remark 3.4 (Required assumptions for the extended system approach)**.** One can easily verify that our main Theorems 3.2 and 3.3 remain valid, as long as the following eight assumptions are satisfied:

- The underlying Hilbert spaces allow for decompositions $X = X_a \oplus X_b \oplus X_c$ and $Y = Y_a \oplus Y_b \oplus Y_c$, where the involved spaces are pairwise orthogonal, see Lemma 2.9.
- For every $u_0 \in X_a$ one obtains $D_\lambda F(\lambda_0, u_0) \in Y_a$, as shown in Lemma 2.12.
- For $u_0 \in X_a$ and $\tau = a, b, c$ the inclusions $LX_\tau \subset Y_\tau$ are satisfied, see Lemma 2.13.
- For $u_0 \in X_a$ and $\tau = b, c$ one has $D_{\lambda u} F(\lambda_0, u_0)[X_\tau] \subset Y_\tau$, see Lemma 2.13.
- For $u_0 \in X_a$ and $\tau = b, c$ we have the inclusion $D_{uu} F(\lambda_0, u_0)[X_\tau, X_a] \subset Y_\tau$, as in Lemma 2.14.
- The orthogonal projector from Definition 2.1 satisfies $P(Y_a) \subset Y_a$, as described in Lemma 2.13.
- The underlying nonlinear operator F is odd, i.e., $F(\lambda, -u) = -F(\lambda, u)$.
- If the kernel function satisfies $\phi_0 \notin X_a$, then $L(X_a) = Y_a$, see Lemma 2.15.

4. Validation of symmetry-induced pitchfork bifurcations.

In this section we combine the theory developed in the earlier parts of the paper with a constructive version of the implicit function theorem to establish the existence of branches of pitchfork bifurcation points which are induced through the cyclic group action defined in (2). For this, we recall a specific branch-validation version of the the constructive implicit function theorem from [23] in Section 4.1, and also describe in detail the system that has to be studied in this context. After that, Section 4.2 demonstrates how the assumptions of the branch validation result can be verified. After briefly outlining our spectral approach to this, we show how our recent paper [20] can be used to determine the necessary norm bound of the inverse of the Fréchet derivative, and we also derive required Lipschitz estimates. Finally, Section 4.3 presents some sample pitchfork curve continuations.

4.1. Establishing branches of pitchfork bifurcation points.

In view of Theorem 3.3 we can establish the existence of a specific pitchfork bifurcation point by proving that the associated extended system (21) has an isolated zero. In our situation, this extended system involves three unknowns — the equilibrium solution u, the kernel function v, and the parameter value λ. Note, however, that the diblock copolymer model has an additional parameter σ, and it was shown in [11, 13] that these bifurcation points combine to form curves parameterized by σ. In the present section, we will explain how a constructive version of the implicit function theorem can be used to rigorously verify these branches in the setting of cyclic symmetries.

For the purposes of this paper, we are interested in finding stationary solutions of the diblock copolymer model which are in fact pitchfork bifurcation points. As equilibrium solutions, they have to be zeros of the nonlinear operator

$$F(\sigma, \lambda, u) = -\Delta(\Delta u + \lambda f(u + \mu)) - \lambda \sigma u \;, \tag{28}$$

which is considered as an operator $F : \mathbb{R} \times \mathbb{R} \times X \to Y$ for the spaces defined in (4), and where in contrast to our earlier usage we also explicitly indicate its dependence on σ. Due to Theorem 3.3, such a zero is a pitchfork bifurcation point at the parameter values σ if it is an isolated zero of the extended operator

$$\mathcal{F}(\sigma, \cdot, \cdot, \cdot) : \left\{ \begin{array}{c} \mathbb{R} \times X_a \times X \to \mathbb{R} \times Y_a \times Y \\ (\lambda, u, v) \mapsto (\ell(v) - 1, F(\sigma, \lambda, u), D_u F(\sigma, \lambda, u)[v]) \end{array} \right. , \tag{29}$$

where the spaces X_a and Y_a were defined in Definitions 2.7 and 2.10, respectively. Notice that we include the explicit dependence on the parameter σ, which is the natural continuation parameter for curves of bifurcation points. In order to simplify the notation going forward, we make use of the abbreviations

$$w = (\lambda, u, v) \in \mathcal{X} = \mathbb{R} \times X_a \times X \qquad \text{and} \qquad \mathcal{Y} = \mathbb{R} \times Y_a \times Y \; . \tag{30}$$

In addition, for applying the generalization of the constructive implicit function theorem from [23] which is taylor-made for branch validation, one needs to verify a number of assumptions through rigorous computer-assisted means. More precisely, one has to accomplish the following:

(H1) Find an approximate zero $w^* = (\lambda^*, u^*, v^*) \in \mathcal{X}$ of $\mathcal{F}(\sigma^*, \cdot) : \mathcal{X} \to \mathcal{Y}$ such that for a real number $\rho > 0$ one has the residual estimate $\|\mathcal{F}(\sigma^*, w^*)\|_{\mathcal{Y}} \le \rho$. This can be done by simply using interval arithmetic, based on a truncated cosine Fourier series representation of the functions $u^* \in X_a$ and $v^* \in X$, see also Remark 2.11 and the discussion in the next section.

(H2) Find a bound $K \ge 0$ such that $\|D_w\mathcal{F}(\sigma^*, w^*)^{-1}\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \le K$. This estimate is by far the technically most involved one, but due to the specific form of $\mathcal{F}$, we can directly quote a result from [20]. This will also be presented in the next section, and the reader can find all the technical details in the cited paper.

(H3) Find Lipschitz bounds for the partial Fréchet derivatives $D_w\mathcal{F}$ and $D_\sigma\mathcal{F}$ of the extended operator $\mathcal{F}$ for all (σ, w) close to (σ^*, w^*) in the following sense. There exist four Lipschitz constants $M_k \ge 0$ for $k = 1, \ldots, 4$, as well as $d_w > 0$ and $d_\sigma > 0$, such that for all pairs $(\sigma, w) \in \mathbb{R} \times \mathcal{X}$ with $\|w - w^*\|_{\mathcal{X}} \le d_w$ and $|\sigma - \sigma^*| \le d_\sigma$ one has

$$\begin{aligned} \|D_w\mathcal{F}(\sigma, w) - D_w\mathcal{F}(\sigma^*, w^*)\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} &\le M_1 \|w - w^*\|_{\mathcal{X}} + M_2 |\sigma - \sigma^*| \,, \\ \|D_\sigma\mathcal{F}(\sigma, u) - D_\sigma\mathcal{F}(\sigma^*, w^*)\|_{\mathcal{Y}} &\le M_3 \|w - w^*\|_{\mathcal{X}} + M_4 |\sigma - \sigma^*| \,, \end{aligned}$$

where $\|\cdot\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})}$ denotes the operator norm in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, and as usual we identify $\mathcal{Y}$ with $\mathcal{L}(\mathbb{R}, \mathcal{Y})$. These estimates are substantially more straightforward than the previous step, and they will be established in the next section.

Under these assumptions, one can then establish the following theorem which guarantees branch segments of zeros of $\mathcal{F}$ parameterized by σ close to the approximate solution (σ^*, w^*). This result is taken from [23, Theorem 5], and it is a simple consequence of the constructive implicit function theorem in [23, Theorem 1]. In fact, the theorem below reduces to the original constructive implicit function theorem in the case $w^{\circledast} = 0$.

Theorem 4.1 (Regular branch segment validation)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and suppose that the nonlinear parameter-dependent operator $\mathcal{F} : \mathbb{R} \times \mathcal{X} \to \mathcal{Y}$ is both Fréchet differentiable and satisfies (H3). Assume that $(\sigma^*, w^*) \in \mathbb{R} \times \mathcal{X}$ satisfies the estimates (H1) and (H2) for some positive constants ρ and K, and let $w^{\circledast} \in \mathcal{X}$ be given with*

$$\left\| D_\sigma\mathcal{F}(\sigma^*, w^*) + D_w\mathcal{F}(\sigma^*, w^*)[w^{\circledast}] \right\|_{\mathcal{Y}} \le \eta \tag{31}$$

for some constant $\eta \ge 0$, which will indicate the slant of the box containing the solution branch. Finally, assume that we have the estimates

$$4K^2\rho M_1 < 1 \qquad \textit{and} \qquad 2K\rho < d_w \; . \tag{32}$$

Then there exist pairs of constants $(\delta_\sigma, \delta_w)$ which satisfy

$$0 < \delta_\sigma \le d_\sigma \;, \qquad 0 < \delta_w \le d_w \;, \qquad \textit{and} \qquad \delta_\sigma \left\| w^{\circledast} \right\|_{\mathcal{X}} + \delta_w \le d_w \;, \tag{33}$$

as well as the two inequalities

$$2KM_1\delta_w + 2K\left(M_1\left\|w^{\circledast}\right\|_{\mathcal{X}} + M_2\right)\delta_\sigma \le 1 \tag{34}$$

and

$$2K\rho + 2K\eta\delta_\sigma + 2K\left(M_1\left\|w^{\circledast}\right\|_{\mathcal{X}}^2 + (M_2+M_3)\left\|w^{\circledast}\right\|_{\mathcal{X}} + M_4\right)\delta_\sigma^2 \le \delta_w\ , \tag{35}$$

and for each pair the following holds. For every parameter $\sigma \in \mathbb{R}$ with $|\sigma - \sigma^*| \le \delta_\sigma$ there exists a unique $w(\sigma) \in \mathcal{X}$ with $\|w(\sigma) - (w^* + (\sigma - \sigma^*)w^{\circledast})\|_{\mathcal{X}} \le \delta_w$, and for which the nonlinear equation $\mathcal{F}(\sigma, w(\sigma)) = 0$ holds. In other words, all solutions of the nonlinear problem $\mathcal{F}(\sigma, w) = 0$ in the slanted set

$$\left\{(\sigma, w) \in \mathbb{R}\times\mathcal{X}\ :\ |\sigma - \sigma^*| \le \delta_\sigma \quad \text{and} \quad \left\|w - \left(w^* + (\sigma-\sigma^*)\,w^{\circledast}\right)\right\|_{\mathcal{X}} \le \delta_w\right\}$$

lie on the branch $\sigma \mapsto w(\sigma)$. In addition, if the mapping $\mathcal{F} : \mathbb{R}\times\mathcal{X} \to \mathcal{Y}$ is k-times continuously Fréchet differentiable, then so is the solution function $\sigma \mapsto w(\sigma)$.

For an illustration of the above theorem we refer the reader to [28, Figure 7]. Note that for the application of this result, and of course also for the verification for the assumptions (H2) and (H3), it is crucial to have the partial Fréchet derivatives of $\mathcal{F}$ at hand. One can easily show that they are given by

$$\begin{aligned} D_w\mathcal{F}(\sigma,w)[\tilde{w}] \;=\;& \Big(\ell(\tilde{v})\ ,\ \ \tilde{\lambda}D_\lambda F(\sigma,\lambda,u) + D_u F(\sigma,\lambda,u)[\tilde{u}]\ , \\ & \quad \tilde{\lambda}D_{\lambda u}F(\sigma,\lambda,u)[v] + D_{uu}F(\sigma,\lambda,u)[v,\tilde{u}] \\ & \qquad + D_u F(\sigma,\lambda,u)[\tilde{v}]\Big) \qquad (36) \\ =\;& \Big(\ell(\tilde{v}),\ \tilde{\lambda}(-\Delta f(u+\mu) - \sigma u) - \Delta(\Delta\tilde{u} + \lambda f'(u+\mu)\tilde{u}) - \lambda\sigma\tilde{u}, \\ & \quad \tilde{\lambda}(-\Delta f'(u+\mu)v - \sigma v) - \Delta(\lambda f''(u+\mu)v\tilde{u}) \\ & \qquad -\Delta(\Delta\tilde{v} + \lambda f'(u+\mu)\tilde{v}) - \lambda\sigma\tilde{v}\Big)\ , \qquad (37) \end{aligned}$$

where we write $w = (\lambda, u, v)$ and $\tilde{w} = (\tilde{\lambda}, \tilde{u}, \tilde{v})$, as well as

$$D_\sigma\mathcal{F}(\sigma,w) \;=\; (0,\ D_\sigma F(\sigma,\lambda,u),\ D_{\sigma u}F(\sigma,\lambda,u)[v]) \;=\; (0,\ -\lambda u,\ -\lambda v)\ . \tag{38}$$

4.2. **Verifying the assumptions for the computer-assisted proofs.** We now address the verification of assumptions (H1) through (H3) of Theorem 4.1. With the exception of the last of these, all of them can be treated as in our previous papers [20, 24, 29]. In view of this, we only provide a short descriptions and leave the details to the cited references.

We begin our discussion by illustrating how (H1) can be established. It was shown in Remark 2.11 that the crucial spaces X_a, X_b, and X_c have straightforward explicit Fourier cosine series representations. Thus, it is natural to find the pitchfork bifurcation point approximation in the form of a truncated series. If we denote the resulting discretization size by $N \in \mathbb{N}$, then we consider the orthogonal projection $P_N : X \to X$ defined via

$$P_N u(x) \;:=\; \sum_{k=1}^{N} a_k\cos(k\pi x) \quad \text{for every} \quad u(x) \;=\; \sum_{k=1}^{\infty} a_k\cos(k\pi x) \quad \text{in } X\ , \tag{39}$$

see also (10). An analogous projection Q_N can also be defined on the image space Y. Thus, one can project both the second and the third component of the extended nonlinear operator $\mathcal{F}$ in (29) onto the spaces $Q_N Y_a$ and $Q_N Y$, respectively, and

	$\mathcal{L}$	$D_w\mathcal{F}(\sigma^*, w^*)$		$\mathcal{L}$	$D_w\mathcal{F}(\sigma^*, w^*)$
Spaces	U_1	X_a	Spaces	V_1	Y_a
	U_2	X		V_2	Y
Arguments	η_1	$\tilde{\lambda}$	Coeff.	α_{11}	0
	v_1	$\tilde{u}$		ℓ_{11}	0
	v_2	$\tilde{v}$		ℓ_{12}	ℓ
Coefficients	β_1	1	Coeff.	β_2	1
	b_{11}	$\Delta f(u^* + \mu) + \sigma^* u^*$		b_{21}	$\Delta f'(u^* + \mu) v^* + \sigma^* v^*$
	c_{11}	$\lambda^* f'(u^* + \mu)$		c_{21}	$\lambda^* f''(u^* + \mu) v^*$
	c_{12}	0		c_{22}	$\lambda^* f'(u^* + \mu)$
	γ_{11}	$\lambda^* \sigma^*$		γ_{21}	0
	γ_{12}	0		γ_{22}	$\lambda^* \sigma^*$

TABLE 4. Reformulating the Fréchet derivative $D_w\mathcal{F}(\sigma^*, w^*)$ as the linear elliptic operator $\mathcal{L}$ defined in equations (40), (41), and (42). In our situation, we have $p = 1$ and $q = 2$, and the spaces, arguments, and coefficients in the respective operator definitions correspond to each other as outlined in the table.

by only allowing arguments $u^* \in P_N X_a$ and $v^* \in P_N X$ one then obtains a finite-dimensional system which can be solved numerically for the solution approximation u^* and the kernel function v^*, at the approximate parameter values λ^*. Notice that the dimension of this system is given by $1 + \lfloor N/n \rfloor + N$, as long as N is larger than n. By choosing appropriate Hilbert space norms on the spaces X and Y as in [20, 24], one can then easily compute an upper bound ρ on the residual based on the Fourier cosine sum representations of u^* and v^*. In fact, for computational convenience we use the norms $\|u\|_X = \|\Delta u\|_{L^2(0,1)}$ and $\|u\|_Y = \|\Delta^{-1} u\|_{L^2(0,1)}$, which are equivalent to the respective standard Sobolev norms on these spaces. Moreover, the rigorous upper bound is established using interval arithmetic, more precisely, the Matlab package INTLAB [21].

We now turn our attention to the hypothesis (H2). The required inverse norm bound for the Fréchet derivative $D_w\mathcal{F}(\sigma^*, w^*)$ presented in (37) can be established directly using the results of [20]. In this paper, we developed a method based on the Neumann series and the construction of a suitable approximate inverse to compute a rigorous bound on the inverse operator norm of certain fourth-order linear elliptic operators which include scalar constraints. More precisely, in [20, Theorem 4.1] we considered a linear operator

$$\mathcal{L} : \mathbb{R}^p \times \prod_{i=1}^{q} U_i \to \mathbb{R}^p \times \prod_{i=1}^{q} V_i \,, \tag{40}$$

where $U_i \subset X$ and $V_i \subset Y$ are suitably chosen closed subspaces, which acts on the argument vector $(\eta_1, \ldots, \eta_p, v_1, \ldots, v_q)$, and whose first p components are given by

$$\sum_{i=1}^{p} \alpha_{ki} \eta_i + \sum_{j=1}^{q} \ell_{kj}(v_j) \qquad \text{for} \qquad k = 1, \ldots, p \,, \tag{41}$$

while the remaining q functional components are

$$-\beta_k \Delta^2 v_k - \sum_{i=1}^{p} b_{ki}\eta_i - \Delta \sum_{j=1}^{q} c_{kj} v_j - \sum_{j=1}^{q} \gamma_{kj} v_j \qquad \text{for} \qquad k = 1, \ldots, q \; . \tag{42}$$

Clearly, the operator $D_w\mathcal{F}(\sigma^*, w^*)$ defined in (37) falls into this category, and the necessary correspondences are collected in Table 4. Thus, we can simply apply this theorem to compute the norm estimate, and we refer the readers to [20, Theorem 4.1] for more details.

As the final step, we have to establish the Lipschitz estimates required in (H3). This can be accomplished similar to our proceeding in [20, 24], so we will keep our discussion as short as possible. For this, we define for every $\ell \in \mathbb{N}_0$ the constant

$$f_{\max}^{(\ell)} := \max_{|\rho| \le \|u^*\|_\infty + \overline{C}_1 d_w} \left| f^{(\ell)}(\rho + \mu) \right| \; , \qquad \text{where} \qquad \overline{C}_1 = 0.149072 \tag{43}$$

denotes the embedding constant from Sobolev's embedding theorem in one space dimension introduced in [24, Lemma 2.3], see also [28]. Furthermore, consider pairs (σ, w) and (σ^*, w^*) in $\mathbb{R} \times \mathcal{X}$, where $\mathcal{X}$ was defined in (30), which satisfy both estimates $|\sigma - \sigma^*| \le d_\sigma$ and $\|w - w^*\| \le d_w$. Then the definition of the operator F in (28) implies the estimates

$$\begin{aligned} &\|D_\lambda F(\sigma, \lambda, u) - D_\lambda F(\sigma^*, \lambda^*, u^*)\|_Y \\ &\qquad \le \quad \|\Delta f(u + \mu) + \sigma u - \Delta f(u^* + \mu) - \sigma^* u^*\|_Y \\ &\qquad \le \quad \|f(u + \mu) - f(u^* + \mu)\|_{L^2} + \|\sigma u - \sigma u^*\|_Y + \|\sigma u^* - \sigma^* u^*\|_Y \\ &\qquad \le \quad \frac{\pi^2 f_{\max}^{(1)} + |\sigma^*| + d_\sigma}{\pi^4} \, \|u - u^*\|_X + \|u^*\|_Y \; |\sigma - \sigma^*| \; , \end{aligned} \tag{44}$$

where we also used the estimates $\|u\|_{L^2} \le \|u\|_X/\pi^2$ and $\|u\|_Y \le \|u\|_X/\pi^4$ for all $u \in X$, see for example [24, Lemma 2.6]. Similarly, one obtains for all $\tilde{u} \in X$ the estimate

$$\begin{aligned} &\|D_u F(\sigma, \lambda, u)[\tilde{u}] - D_u F(\sigma^*, \lambda^*, u^*)[\tilde{u}]\|_Y \\ &\qquad \le \quad \|\Delta(\lambda f'(u + \mu)\tilde{u} - \lambda^* f'(u^* + \mu)\tilde{u})\|_Y + |\lambda\sigma - \lambda^*\sigma^*| \, \|\tilde{u}\|_Y \\ &\qquad \le \quad |\lambda| \, \|f'(u + \mu)\tilde{u} - f'(u^* + \mu)\tilde{u}\|_{L^2} + |\lambda - \lambda^*| \, \|f'(u^* + \mu)\tilde{u}\|_{L^2} \\ &\qquad\qquad\qquad + |\lambda - \lambda^*| \, \frac{|\sigma|}{\pi^4} \, \|\tilde{u}\|_X + |\sigma - \sigma^*| \, \frac{|\lambda^*|}{\pi^4} \, \|\tilde{u}\|_X \\ &\qquad \le \quad |\lambda| \, f_{\max}^{(2)} \, \|u - u^*\|_\infty \|\tilde{u}\|_{L^2} + |\lambda - \lambda^*| \, \|f'(u^* + \mu)\|_\infty \, \|\tilde{u}\|_{L^2} \\ &\qquad\qquad\qquad + |\lambda - \lambda^*| \, \frac{|\sigma|}{\pi^4} \, \|\tilde{u}\|_X + |\sigma - \sigma^*| \, \frac{|\lambda^*|}{\pi^4} \, \|\tilde{u}\|_X \; , \end{aligned}$$

which in turn implies

$$\begin{aligned} &\|D_u F(\sigma, \lambda, u) - D_u F(\sigma^*, \lambda^*, u^*)\|_{\mathcal{L}(X,Y)} \\ &\qquad \le \quad \frac{\pi^2 \|f'(u^* + \mu)\|_\infty + |\sigma^*| + d_\sigma}{\pi^4} \, |\lambda - \lambda^*| \\ &\qquad\qquad\qquad + \frac{\overline{C}_1 f_{\max}^{(2)}(|\lambda^*| + d_w)}{\pi^2} \, \|u - u^*\|_X + \frac{|\lambda^*|}{\pi^4} \, |\sigma - \sigma^*| \; . \end{aligned} \tag{45}$$

We now start estimating the two terms which remain in the last component of the operator $\mathcal{F}$. On the one hand, we have

$$\|D_{\lambda u} F(\sigma, \lambda, u)[v] - D_{\lambda u} F(\sigma^*, \lambda^*, u^*)[v^*]\|_Y \le$$

$$
\begin{aligned}
&\le \|\Delta(f'(u+\mu)v - f'(u^*+\mu)v^*)\|_Y + \|\sigma v - \sigma^* v^*\|_Y \\
&\le \|f'(u+\mu)v - f'(u^*+\mu)v^*\|_{L^2} + |\sigma|\,\|v-v^*\|_Y + \|v^*\|_Y\,|\sigma-\sigma^*| \\
&\le \|f'(u+\mu)v - f'(u+\mu)v^*\|_{L^2} + \|f'(u+\mu)v^* - f'(u^*+\mu)v^*\|_{L^2} \\
&\qquad + \frac{|\sigma|}{\pi^4}\,\|v-v^*\|_X + \|v^*\|_Y\,|\sigma-\sigma^*| \\
&\le \frac{f^{(1)}_{\max}}{\pi^2}\,\|v-v^*\|_X + \frac{f^{(2)}_{\max}\,\|v^*\|_\infty}{\pi^2}\,\|u-u^*\|_X \\
&\qquad + \frac{|\sigma|}{\pi^4}\,\|v-v^*\|_X + \|v^*\|_Y\,|\sigma-\sigma^*| \\
&\le \frac{f^{(2)}_{\max}\,\|v^*\|_\infty}{\pi^2}\,\|u-u^*\|_X + \frac{\pi^2 f^{(1)}_{\max} + |\sigma^*| + d_\sigma}{\pi^4}\,\|v-v^*\|_X \\
&\qquad + \|v^*\|_Y\,|\sigma-\sigma^*|\,, \qquad (46)
\end{aligned}
$$

while on the other hand one obtains the estimate

$$
\begin{aligned}
&\|D_{uu}F(\sigma,\lambda,u)[v,\tilde u] - D_{uu}F(\sigma^*,\lambda^*,u^*)[v^*,\tilde u]\|_Y \\
\le\;& \|\Delta(\lambda f''(u+\mu)v\tilde u - \lambda^* f''(u^*+\mu)v^*\tilde u)\|_Y \\
\le\;& \|\lambda f''(u+\mu)v\tilde u - \lambda f''(u^*+\mu)v\tilde u\|_{L^2} \\
&+ \|\lambda f''(u^*+\mu)v\tilde u - \lambda^* f''(u^*+\mu)v^*\tilde u\|_{L^2} \\
\le\;& |\lambda|\, f^{(3)}_{\max}\,\|u-u^*\|_\infty\,\|v\|_\infty\,\|\tilde u\|_{L^2} + \|\lambda f''(u^*+\mu)v\tilde u - \lambda f''(u^*+\mu)v^*\tilde u\|_{L^2} \\
&+ \|\lambda f''(u^*+\mu)v^*\tilde u - \lambda^* f''(u^*+\mu)v^*\tilde u\|_{L^2} \\
\le\;& |\lambda|\, f^{(3)}_{\max}\,\|u-u^*\|_\infty\,\|v\|_\infty\,\|\tilde u\|_{L^2} + |\lambda|\,\|f''(u^*+\mu)\|_\infty\,\|v-v^*\|_\infty\,\|\tilde u\|_{L^2} \\
&+ |\lambda-\lambda^*|\,\|f''(u^*+\mu)v^*\|_\infty\,\|\tilde u\|_{L^2}\,,
\end{aligned}
$$

which in turn implies

$$
\begin{aligned}
&\|D_{uu}F(\sigma,\lambda,u)[v,\cdot] - D_{uu}F(\sigma^*,\lambda^*,u^*)[v^*,\cdot]\|_{\mathcal{L}(X,Y)} \\
\le\;& \frac{\|f''(u^*+\mu)v^*\|_\infty}{\pi^2}\,|\lambda-\lambda^*| + \frac{\overline{C}_1 f^{(3)}_{\max}(|\lambda^*|+d_w)(\|v^*\|_\infty + \overline{C}_1 d_w)}{\pi^2}\,\|u-u^*\|_X \\
&+ \frac{\overline{C}_1\|f''(u^*+\mu)\|_\infty(|\lambda^*|+d_w)}{\pi^2}\,\|v-v^*\|_X\,. \qquad (47)
\end{aligned}
$$

Altogether, we have established the estimates

$$
\begin{aligned}
\|D_\lambda F(\sigma,\lambda,u) - D_\lambda F(\sigma^*,\lambda^*,u^*)\| &\le c_1\|u-u^*\| + c_2|\sigma-\sigma^*|\,, \\
\|D_u F(\sigma,\lambda,u) - D_u F(\sigma^*,\lambda^*,u^*)\| &\le c_3|\lambda-\lambda^*| + c_4\|u-u^*\| \\
&\qquad + c_5|\sigma-\sigma^*|\,, \\
\|D_{\lambda u} F(\sigma,\lambda,u)[v] - D_{\lambda u} F(\sigma^*,\lambda^*,u^*)[v^*]\| &\le c_6\|u-u^*\| + c_7\|v-v^*\| \\
&\qquad + c_8|\sigma-\sigma^*|\,, \\
\|D_{uu} F(\sigma,\lambda,u)[v,\cdot] - D_{uu} F(\sigma^*,\lambda^*,u^*)[v^*,\cdot]\| &\le c_9|\lambda-\lambda^*| + c_{10}\|u-u^*\| \\
&\qquad + c_{11}\|v-v^*\|\,,
\end{aligned}
$$

where the Lipschitz constants c_k can be inferred from equations (43) through (47), and we dropped the subscripts indicating the specific norms. After these preparations, one can now easily establish the estimates in (H3). For this, we write the nonlinear operator $\mathcal{F}$ in component form as $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. Then we clearly have

$$
D_w\mathcal{F}_1(\sigma,w)[\tilde w] - D_w\mathcal{F}_1(\sigma^*,w^*)[\tilde w] = 0\,,
$$

while for the second component one obtains

$$\begin{aligned}
&\|D_w\mathcal{F}_2(\sigma,w)[\tilde{w}] - D_w\mathcal{F}_2(\sigma^*,w^*)[\tilde{w}]\|_Y \\
&\quad\le \|D_\lambda F(\sigma,\lambda,u) - D_\lambda F(\sigma^*,\lambda^*,u^*)\|_Y|\tilde{\lambda}| \\
&\qquad\qquad + \|D_u F(\sigma,\lambda,u)[\tilde{u}] - D_u F(\sigma^*,\lambda^*,u^*)[\tilde{u}]\|_Y \\
&\quad\le (c_1\|u-u^*\|_X + c_2|\sigma-\sigma^*|)\,|\tilde{\lambda}| \\
&\qquad\qquad + (c_3|\lambda-\lambda^*| + c_4\|u-u^*\|_X + c_5|\sigma-\sigma^*|)\,\|\tilde{u}\|_X \\
&\quad\le (c_3|\lambda-\lambda^*| + (c_1+c_4)\|u-u^*\|_X + (c_2+c_5)|\sigma-\sigma^*|)\,\|\tilde{w}\|_{\mathcal{X}} \\
&\quad\le \left(\sqrt{c_3^2+(c_1+c_4)^2}\,\|w-w^*\|_{\mathcal{X}} + (c_2+c_5)|\sigma-\sigma^*|\right)\|\tilde{w}\|_{\mathcal{X}} \; ,
\end{aligned}$$

and similarly for the third component

$$\begin{aligned}
&\|D_w\mathcal{F}_3(\sigma,w)[\tilde{w}] - D_w\mathcal{F}_3(\sigma^*,w^*)[\tilde{w}]\|_Y \\
&\quad\le (c_6\|u-u^*\|_X + c_7\|v-v^*\|_X + c_8|\sigma-\sigma^*|)\,|\tilde{\lambda}| \\
&\qquad\qquad + (c_9|\lambda-\lambda^*| + c_{10}\|u-u^*\|_X + c_{11}\|v-v^*\|_X)\,\|\tilde{u}\|_X \\
&\qquad\qquad + (c_3|\lambda-\lambda^*| + c_4\|u-u^*\|_X + c_5|\sigma-\sigma^*|)\,\|\tilde{v}\|_X \\
&\quad\le ((c_3+c_9)|\lambda-\lambda^*| + (c_4+c_6+c_{10})\|u-u^*\|_X \\
&\qquad\qquad + (c_7+c_{11})\|v-v^*\|_X + (c_5+c_8)|\sigma-\sigma^*|)\,\|\tilde{w}\|_{\mathcal{X}} \\
&\quad\le \Big(\sqrt{(c_3+c_9)^2+(c_4+c_6+c_{10})^2+(c_7+c_{11})^2}\,\|w-w^*\|_{\mathcal{X}} \\
&\qquad\qquad + (c_5+c_8)|\sigma-\sigma^*|\Big)\,\|\tilde{w}\|_{\mathcal{X}} \; .
\end{aligned}$$

If we now define the constants M_1 and M_2 as

$$M_1 = \sqrt{2\max\{c_3^2+(c_1+c_4)^2,\ (c_3+c_9)^2+(c_4+c_6+c_{10})^2+(c_7+c_{11})^2\}}, \tag{48}$$

$$M_2 = \sqrt{2}\,\max\{c_2+c_5\,,\ c_5+c_8\}\ , \tag{49}$$

then one immediately obtains

$$\|D_w\mathcal{F}(\sigma,w)[\tilde{w}] - D_w\mathcal{F}(\sigma^*,w^*)[\tilde{w}]\|_{\mathcal{Y}} \le (M_1\|w-w^*\|_{\mathcal{X}} + M_2|\sigma-\sigma^*|)\,\|\tilde{w}\|_{\mathcal{X}}\ ,$$

i.e., the first estimate in (H3) holds. Furthermore, in view of (38) we have the estimate

$$\begin{aligned}
&\|D_\sigma\mathcal{F}(\sigma,w) - D_\sigma\mathcal{F}(\sigma^*,w^*)\|_{\mathcal{Y}} \\
&\quad\le \left(\|\lambda u - \lambda^* u^*\|_Y^2 + \|\lambda v - \lambda^* v^*\|_Y^2\right)^{1/2} \\
&\quad\le \sqrt{2}\,\left(\|\lambda u - \lambda u^*\|_Y^2 + \|\lambda u^* - \lambda^* u^*\|_Y^2 + \|\lambda v - \lambda v^*\|_Y^2 + \|\lambda v^* - \lambda^* v^*\|_Y^2\right)^{1/2} \\
&\quad\le \sqrt{2}\,\Big(|\lambda|^2\,\|u-u^*\|_Y^2 + \|u^*\|_Y^2\,|\lambda-\lambda^*|^2 \\
&\qquad\qquad + |\lambda|^2\,\|v-v^*\|_Y^2 + \|v^*\|_Y^2\,|\lambda-\lambda^*|^2\Big)^{1/2} \\
&\quad\le \sqrt{2}\,\max\left\{\frac{|\lambda^*|+d_w}{\pi^4}\,,\ \sqrt{\|u^*\|_Y^2+\|v^*\|_Y^2}\right\}\|w-w^*\|_{\mathcal{X}}\ ,
\end{aligned}$$

and therefore the second estimate in (H3) is satisfied with

$$M_3 = \sqrt{2}\,\max\left\{\frac{|\lambda^*|+d_w}{\pi^4}\,,\ \sqrt{\|u^*\|_Y^2+\|v^*\|_Y^2}\right\} \quad\text{and}\quad M_4 = 0\,. \tag{50}$$

This completes the verification of the assumptions of the regular branch segment validation result in Theorem 4.1.

n	λ	N	τ	K	M_1	d_w
5	115.69	178	0.49917	73.453	54.859	1.2408e-04
7	315.57	670	0.29987	154.78	150.85	2.1415e-05
4	336.05	874	0.49972	503.86	253.14	3.9201e-06
6	769.12	2000	0.29979	829.58	636.9	9.4642e-07

TABLE 5. Validation parameters for the validated solutions.

4.3. **Sample computational validations.** In this section, we implement the techniques listed above in order to computationally validate the first two solutions shown in Figure 1, and the first two solutions shown in Figure 2 for the fixed parameter value $\sigma = 6$. The computed validation parameters are given in Table 5. For larger λ values, the computationally necessary value of N becomes extremely large if one uses the model equations in their original unmodified form. Notice that this is unavoidable, as the number of modes required to represent the solutions increases quickly with increasing λ. In addition, in this limit the equation becomes closer to singular, and therefore one fully expects that numerical approaches become significantly more difficult.

Nevertheless, our focus in this paper is on the analytical background for a new type of symmetry-breaking pitchfork bifurcation. For the sake of space, we therefore do not address the computationally heavy methods needed to refine this method, and choose to address this numerical machinery in a future work. The following are two specific techniques which we plan to incorporate in future.

In order to improve the validation and to address the validation of further solutions with larger n values and substantially larger λ values at bifurcation, we would need to include preconditioning in our validation, which essentially amounts to rescalings in the underlying partial differential equation. In particular, since both λ versus (u, v), and the components of function $\mathcal{F}_e$, occur on extremely different length scales, one would expect to see rather substantial stiffness in the computation of both the Lipschitz constants and the bound K. See for example the improvement in the method due to preconditioning in the recent paper [12]. After preconditioning is established, we would further be able to consider the case of varying σ, and create a validated continuation method to find the curve of bifurcation points in the two-parameter family. In [12], we developed a validated pseudo-arclength continuation method. In future work, we intend to adapt this method to the current setting.

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Received December 2022; revised May 2023; early access June 2023.