Hyperbolic Sets for Noninvertible Maps and Relations

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ABSTRACT

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by Evelyn Sander

This thesis presents a theory of hyperbolic structures and dynamics of smooth noninvertible maps and relations. In this context, it includes a new proof of the stable manifold theorem for fixed points, the shadowing lemma, and a version of the stable manifold theorem for hyperbolic sets. It also gives a description of some of the behavior of transverse homoclinic orbits for noninvertible maps and relations.

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Chapter 1

Introduction

1.1 Preliminary remarks

The mathematical theory of dynamical systems is the study of systems governed by a consistent set of laws over time. Even very simple systems can result in highly complicated behavior, so rather than trying to solve such systems explicitly, we study them qualitatively.

Dynamical systems originally focussed on the behavior of continuous systems, governed by differential equations. The approach has been applied to a number of famous problems, such as Poincaré's seminal study of the motion of the planets under the laws of Newtonian mechanics, the Lotka-Volterra model for the rate of change in the population of animal species competing for resources, and the Lorenz equations for fluid flow in the atmosphere. For processes modelled by differential equations, there are often related maps which reflect the dynamics of the system, the principal example of which is the Poincaré map. Looking at the Poincaré map makes a dynamical system conceptually simpler, since it reduces the dimension of the space.

Motivated by this need to understand the dynamics of the Poincaré map, in the 1960's there was a great surge of interest in the study of diffeomorphisms in their own right. The theory of dynamics of diffeomorphisms has come to be very useful in applications independent of the theory of continuous dynamics.

Not all iterated maps arising in natural systems have the property that time is reversible. In other words, there may be multiple points which map to the same point under iteration. This is evident in one-dimensional dynamics; for example, the frequently studied logistic map $x_{n+1} = \mu x_n(1 - x_n)$ is noninvertible. Despite the potential usefulness of such a theory, there has been little study of iterated noninvertible maps in dimensions greater than one. We are currently experiencing expansion of the field to include noninvertible maps. This thesis generalizes the ideas of hyperbolic sets to the noninvertible setting.

Rather than restricting only to noninvertible maps, in this thesis we work in the setting of *relations*. Relations are maps which can be not only noninvertible, having non-unique preimage, but also multivalued, with non-unique image. Here is a motivation for working in this more general setting: If f is a diffeomorphism, then so is f^{-1}. Thus forward and backward iteration are symmetric concepts. This symmetry is used extensively in the study of diffeomorphisms. In contrast, the inverse of a smooth noninvertible map is not map; there is a qualitative distinction between going forward and going backward. On the other hand, the inverse of a relation is another relation. Thus the symmetry of iteration is restored. See Definition 3.4 for the formal definition of a relation. Section 3.3 gives further definitions and framework.

1.2 Previous theory and applications

There have been many works studying specific examples of noninvertible maps. For example, the work of Aronson, Chory, Hall, and McGehee [3] describe a noninvertible map in a simple population model. Hale and Lin [11] describe time one maps of delay equations, which may be noninvertible. Several works of Adomaitis, Frouzakis, and Kevrekidis [7, 8, 1] describe the dynamics of noninvertible maps arising in control theory algorithms. Rico-Martínez, Kevrekidis, and Adomaitis [16] study a noninvertible maps arising in neural networks. Finally, Lorenz [14] shows the behavior of a noninvertible map arising from a large time step when applying an iterated difference method to approximate an ordinary differential equation. Chapter 2 describes these applications in more detail. In addition to maps arising in the modelling of natural systems, many abstract examples of noninvertible maps of the plane can be found in [9, 10], and other related works.

In comparison to the wealth of specific examples, there are relatively few attempts to classify the general theory of iterated noninvertible maps. Of particular note are the works of Hale and Lin [11] and Marotto [15] on homoclinic orbits, the work of McGehee in the context of iterated relations [18], and the work of Steinlein and Walther [27, 26] for hyperbolic sets in Banach spaces.

1.2.1 The noninvertible stable manifold theorem

How does one classify the dynamics of iterated maps? The simplest orbits are fixed points, which themselves constitute an entire orbit. The stable manifold theorem is a beautiful characterization of behavior near a fixed point at which the derivative has no eigenvalues of norm one, called a *hyperbolic* fixed point. It says that for a smooth map, points locally converging to a hyperbolic fixed point form a smooth manifold. This is true both for points converging forwards in time, forming the stable manifold, and also for points converging backwards in time, forming the unstable manifold.

The unstable manifold of a diffeomorphism is the same as the stable manifold of its inverse. Thus the stable and unstable versions of the manifold theorem are symmetric statements with symmetric proofs. However, since the inverse of a noninvertible map is no longer a map, there is no apparent symmetry between the noninvertible stable and unstable manifolds. Chapter 3 gives a new proof of the noninvertible stable manifold theorem from joint work with Richard McGehee [17]. By considering relations, the stable and unstable manifolds retain the symmetry of the diffeomorphism case.

1.3 Hyperbolic sets, stable manifolds, and the shadowing lemma

By results from spectral theory, the tangent space of a hyperbolic fixed point is the direct sum of a *stable* subspace and an *unstable* subspace. The derivative is eventually contracting on the stable subspace and eventually expanding on the unstable subspace. This notion of hyperbolicity at fixed points generalizes invariant sets of diffeomorphisms. Namely, at each point in an invariant set with *hyperbolic structure*, there is a continuous splitting of the tangent space into stable and unstable subspaces. These subspaces map invariantly under the derivative; on the stable subspaces the derivative is eventually uniformly contracting, and on the unstable subspaces the derivative is eventually uniformly expanding.

In order to define hyperbolic sets for noninvertible maps, we must reformulate the assumption that a hyperbolic splitting maps invariantly under the derivative. Even a hyperbolic fixed point of a linear map can lose the strict invariance property for the stable subspace; the subspace collapses under the action of a singular linear map. In addition to this problem with the stable set, the assumption that the expanding subspaces map invariantly is also too strong [27]; since there can be multiple points mapping to the same point, it is too restrictive to insist on the existence of a continuous splitting which maps invariantly under the derivative.

From these examples, we see that the generalization of the definition of hyperbolic sets is quite delicate. The ideas from our proof of the stable manifold theorem give a way to define hyperbolic sets for smooth relations. Again, looking at relations preserves symmetry of forwards and backwards iteration in hyperbolic sets.

For diffeomorphisms, the stable manifold theorem for fixed points generalizes to a theorem for hyperbolic sets. Namely, given a point p in a hyperbolic set, the set of points with forward iterates converging to forward iterates of p locally form a smooth stable manifold. In addition, the set of points with backward iterates converging to backward iterates of p locally form a smooth unstable manifold. For noninvertible maps, it is no longer possible to talk about an unstable manifold to a point, since there may be many points mapping to the same point, and thus many unstable manifolds. However, restricting to forward and backwards orbits, a similar theorem still holds for noninvertible maps and relations, as described in Chapter 4.

Another result for hyperbolic sets for diffeomorphisms is the shadowing lemma of Bowen [5]. It pertains to *pseudo-orbits*; these are sequences of points such that the image of each point in the sequence is no more than a small previously specified distance from the next point in the sequence. The shadowing lemma says that near a compact invariant hyperbolic set, every bi-infinite pseudo-orbit has a unique nearby exact orbit. Thus small mistakes in iteration do not effect the qualitative picture of the dynamics of the map. Chapter 4 contains a proof of the shadowing lemma for smooth noninvertible maps and relations.

1.4 Transverse homoclinic orbits

An important application of the shadowing lemma is to describe behavior of transverse homoclinic orbits. Homoclinic orbits are orbits which converge to a hyperbolic fixed point both forwards and backwards; transverse means that at some point in the orbit, the stable and unstable manifolds intersect transversally. For a diffeomorphism, the closure of a transverse homoclinic orbit to a hyperbolic fixed point is a compact hyperbolic set. The stable and unstable subspaces of the hyperbolic splitting are the subspaces tangent to the stable and the unstable manifolds. The shadowing lemma implies that near a transverse homoclinic orbit to a fixed point, there is an invariant set upon which an iterate of the diffeomorphism is topologically conjugate to a subshift of finite type. This sort of complicated behavior is often referred to as a *homoclinic tangle.*

Since there is a noninvertible shadowing lemma one would think that transverse homoclinic orbits of noninvertible maps would be similar to homoclinic tangles for diffeomorphisms; but this is not true, because in general the closure of a transverse homoclinic orbit is not a hyperbolic set. Chapter 5 contains an example of a map with a transverse homoclinic orbit near which there is no recurrent behavior. The chapter also contains appropriate conditions for noninvertible maps and relations to have a homoclinic tangle.

Chapter 2

Applications

This chapter describes a few applications in which it is appropriate to study iterated smooth noninvertible maps and relations.

2.1 Difference methods

Non-autonomous differential equations of the form

$$\dot{x} = F(x),$$

always have invertible solutions. However, numerical approximations are often noninvertible for too large a time step. For example, this is true in the simple case of the Euler difference approximation:

$$x_{n+1} = x_n + \tau F(x_n).$$

Lorenz describes this phenomenon [14] in the case $F(x) = x - x^2$. In this case, the Euler approximation gives $x_{n+1} = (1+\tau)x_n - \tau x_n^2$, a version of the quadratic equation.

Lorenz also describes the nature of solutions to a two-dimensional version of the Lorenz equations. The way in which the approximation fails to behave like the original solution depends on the noninvertibility of the map. The noninvertible nature of an approximation may result in a qualitative change in behavior from the true solution. For example, it may result in a smooth invariant circle losing its smoothness, containing overlapping loops, or containing disconnected components. For a bounded invertible map, iterates inside an invariant circle all remain inside the circle under iteration. Thus for differential equations, invariant circles restrict movement. For noninvertible maps, this is no longer true. The invariant circles for the examples in Lorenz's paper [14] display these kinds of noninvertible behavior.

It is theoretically possible to make the time step small enough that there is a clear choice of inverse for a difference approximation. However, in applications such as meteorology, in which there are many variables and multiple time scales, it is infeasible to make the time step sufficiently small.

2.2 Adaptive control

Adaptive control systems automatically control processes using periodic feedback information. One tests the *output state* of a system at regular intervals, using this information to determine the next *input*. The goal is to reach a preset ideal output, called the *set point*. This process is associated with a map from output and input at time n to output and input at time $n+1$. Ideally, we would like to find an algorithm for controlling the input so as to map any output at time n to the set point at time $n+1$. In other words, the ideal control map is noninvertible. Thus an iterated map modelling an adaptive control algorithm is inherently noninvertible.

A simple example, described in [7, 8, 1], considers a constantly mixing tank with a stream of water and a concentrated dye solution flowing in, and a well mixed solution flowing out. The controlled input is the rate at which concentrated solution of dye flows into the tank. The output, measured with a photometer, is the concentration of dye in the tank after each discrete time interval; the set point is some desired concentration of dye. The map governing this situation is noninvertible. The papers cited above show experimental data verifying the noninvertible nature of the model. For example, they show that that the basin of attraction of the set point consists of a series of disconnected components.

2.3 Delay equations

A delay differential equation describes a vector field depending on both current and previous values of a solution. For example, an equation of the form

$$\dot{x}(t) = F(x(t-1)).$$

A periodic solution for a delay equation defines a Poincaré map. However, such a map will not in general be invertible. The works of Steinlein and Walther [27, 26] and Hale and Lin [11] study the nature of such noninvertible maps. Poincaré maps from delay equations are infinite dimensional; only finite dimensional approximations can be studied with the theory developed in this thesis.

2.4 Delayed regulation population map

A standard model of population is of the form

$$P_{n+1} = RP_n,$$

where P_n is the population at time n. According to Maynard Smith **Mathematical Ideas in Biology** [25], there are practical situations in which R depends on P_{n-1}. The reproduction rate for a herbivorous species may depend most strongly on the amount of vegetation eaten in the previous year. A paper of Aronson,

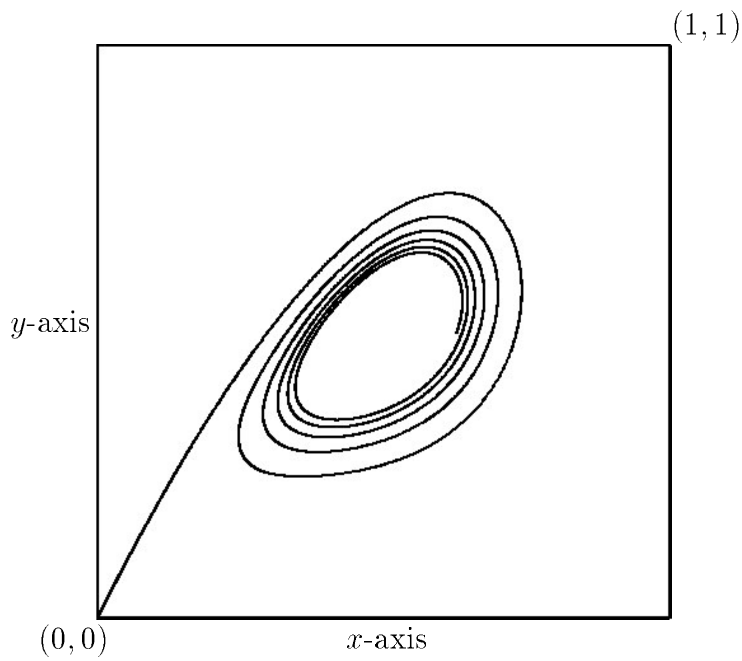

Figure 2.1: Unstable manifold of the origin for the delayed regulation map, $a = 2.0$.

Chory, Hall, and McGehee [3] analyzes the following version of this population model, which they call the delayed regulation model:

$$P_{n+1} = aP_n(1 - P_{n-1}).$$

Making the change of variable $x_n = P_n$ and $y_n = P_{n-1}$, we get

$$\begin{array}{rcl} x_{n+1} & = & y_n \\ y_{n+1} & = & ay_n(1 - x_n) \end{array}.$$

This is clearly a noninvertible map, since the entire x-axis gets mapped to the origin. Thus the fixed point at the origin has the x-axis as its local stable manifold. For small values of a (< 2.27), iterates of points on the unstable manifold to the origin converge to an attracting circle, as shown in the Figure 2.1.

When a is approximately 2.27, the stable and unstable manifolds appear tangent, as shown in Figure 2.2. Finally, for $a > 2.27$, the stable and unstable manifolds intersect transversally, as shown in Figure 2.3. This transverse homoclinic orbit will be used as a principle example of a hyperbolic set for a noninvertible map.

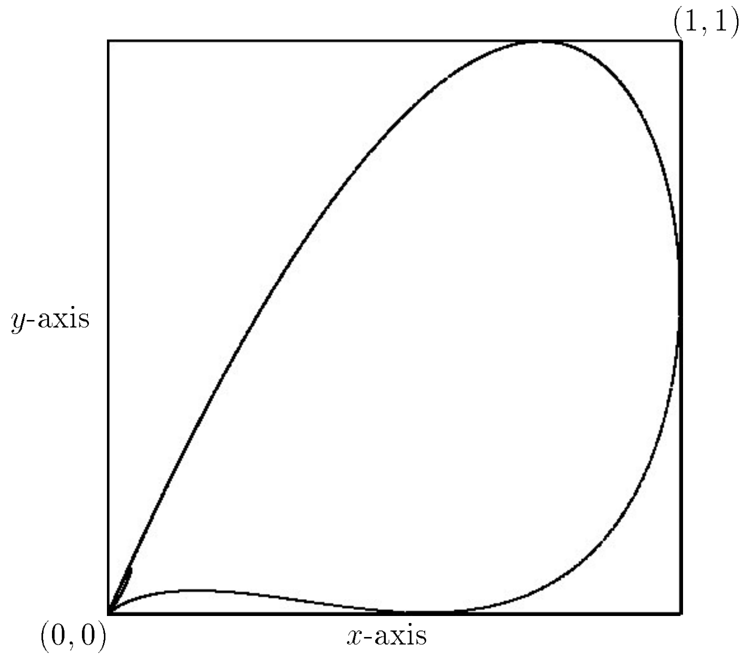

Figure 2.2: Unstable manifold of the origin for the delayed regulation map, $a = 2.27$. The unstable manifold is tangent to the x-axis, the local stable manifold.

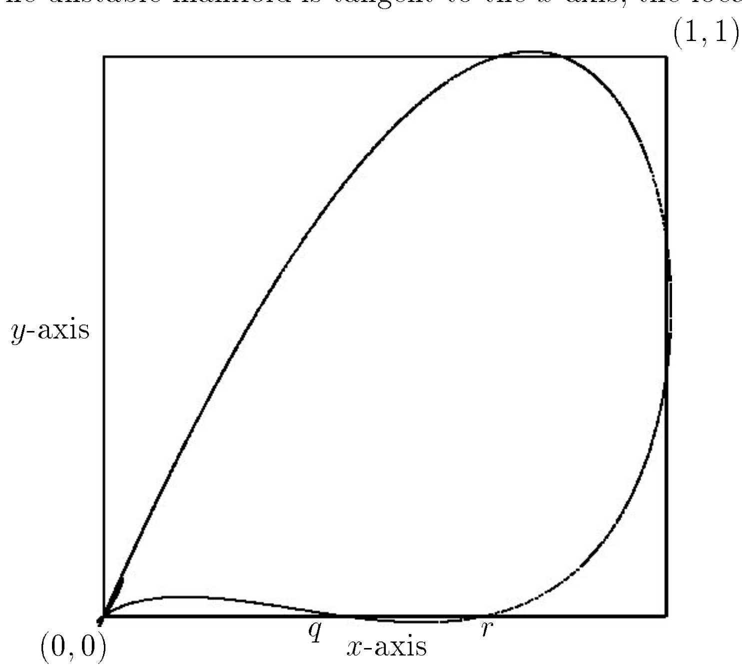

Figure 2.3: Unstable manifold of the origin for the delayed regulation map, $a = 2.28$. The unstable manifold intersects the stable manifold transversally at points q and r on the x-axis.

2.5 Multivalued maps

In addition to applications for noninvertible maps, there are also some applications for iterated multivalued maps. For example, Barnsley uses *hyperbolic iterated function systems* as a means of creating fractals [4]. These function systems are simple examples of multivalued relations.

Definition 2.1 (Iterated function systems) *An iterated function system is a complete metric space with a finite set of contraction mappings* $\{w_n, n = 1, \ldots, N\}$.

Of interest is the dynamics of the multivalued map $\cup_{n=1}^{N} w_n$. By the definitions in Chapter 3, the graph of each such multivalued map is a relation with hyperbolic structure. Thus it is possible to describe the dynamics of such a map using results for hyperbolic sets presented here.

Chapter 3

The Stable Manifold Theorem

This chapter contains of a new proof of the local stable manifold theorem for hyperbolic fixed points of smooth relations. This proof shows that the local stable and unstable manifolds are projections of a relation obtained as a limit of the graphs of the iterates of the original relation.

In contrast to the local stable manifold theorem, the global stable manifold theorem does not hold in the noninvertible case. The last section of this chapter describes a variety of ways in which the theorem can fail.

3.1 Background

The stable manifold theorem states that near a hyperbolic fixed point p, points with forward orbit converging to p and points with backwards orbit converging to p are both smooth manifolds. This is a standard theorem for diffeomorphisms [12, 19], and is also known to be true for noninvertible maps [23, 11]. This chapter gives a new proof of the stable manifold theorem, a joint work with Richard McGehee first presented in [17]. The proof is in the context of "smooth relations" [2, 18], a generalization which includes as special cases hyperbolic fixed points of both invertible and noninvertible maps.

The proof presented here is not merely an effort to generalize the standard theorem to the case of relations. Looking at relations restores to the noninvertible case the symmetry between the stable and unstable manifolds as is seen in the diffeomorphism case. In addition, it provides a new geometric way of looking at the local stable and unstable manifolds of a map; namely, they are both projections of an object one can think of as the "infinite iterate" of the graph of the map.

The key to this new proof is that rather than looking at stable and unstable manifolds as subsets of the state space, we view them as projections of a smooth manifold in higher dimensions arising from the graph of the original map. More precisely, near a hyperbolic fixed point, the graph of a map and the graphs of its iterates can be expressed in an appropriate coordinate system as graphs of smooth contractions. The limit of these contractions exists and is smooth. The graph of this limit projects to the stable and unstable manifolds.

The derivative of a smooth map on R^n at a hyperbolic fixed point has no eigenvalues on the unit circle. Thus locally, in coordinates given by the stable and unstable directions X and Y, a map can be expressed as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Ax + g_1(x,y) \\ By + g_2(x,y) \end{pmatrix}, \tag{3.1}$$

where x and y are vectors in X and Y, (x', y') being the iterate of (x, y), A and B matrices with $|A| < 1, |B^{-1}| < 1$, and g_1 and g_2 are Lipschitz with small Lipschitz constant.

By the Implicit Function Theorem, we can locally change to a skewed coordinate system such that in these new coordinates, we have a local contraction. Namely, we can write:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} Ax + \hat{g}_1(x,y') \\ B^{-1}y' + \hat{g}_2(x,y') \end{pmatrix}. \tag{3.2}$$

$\hat{g}_1$ and $\hat{g}_2$ are again Lipschitz with small Lipschitz constant.

The proof presented here capitalizes on the fact that the map and all its iterates are local contractions when written in this skewed coordinate system. Before presenting the proof, we illustrate the ideas with some simple examples.

3.2 Some simple examples

Example 3.1 Consider the graph of the following linear diffeomorphism f on R^2 with hyperbolic fixed point $(0,0)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{for } 0 < a < 1 < b. \tag{3.3}$$

Since the x-axis and the y-axis are respectively the one-dimensional stable and unstable directions, we choose them to be the directions X and Y respectively in the skewed coordinates. Call the new function resulting from writing f in skewed coordinates ϕ_1. It is written as follows:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} x \\ y' \end{pmatrix}. \tag{3.4}$$

The k^{th} iterate of the original map is

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.5}$$

Writing the k^{th} iterate in skewed coordinates, gives the following function ϕ_k. Note that ϕ_k is found by looking at f^k and *not* by iterating ϕ_1, although in this case both methods give the same answer.

$$\begin{pmatrix} x_k \\ y \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 0 & \frac{1}{b^k} \end{pmatrix} \begin{pmatrix} x \\ y_k \end{pmatrix}. \tag{3.6}$$

Consider the limit of the ϕ_k; it exists and is equal to the map which is identically zero; explicitly, $\lim_{k\to\infty}\phi_k$ is the following map in skewed coordinates on R^2:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y' \end{pmatrix}. \tag{3.7}$$

Notice that this limit map in the skewed coordinate system does not correspond to a function in the original coordinates. However, we can gain information about the stable and unstable manifolds from its graph. Namely, the projection of the graph to the xy- plane is the x-axis, the stable manifold. The projection of the graph to the $x'y'$- plane is y'-axis, the unstable manifold.

Example 3.2 The trick in Example 3.1 still works if the linear map is noninvertible; i.e. if $a = 0$. The map becomes:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ by \end{pmatrix}, \qquad \text{for } 1 < b, \tag{3.8}$$

which can still be expressed in the same skewed coordinates as before:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{b}y' \end{pmatrix}. \tag{3.9}$$

The limit of the ϕ_k, the k^{th} iterate written in skewed coordinates, is the same as before. Indeed, the stable and unstable manifolds are once again the x-axis and y-axis respectively.

Example 3.3 If we allow the stretching term b in Example 3.1 to increase without bound, the graph of the map of f converges to $\{(u, 0, au, v) : (u, v) \in R^2\}$. This is no longer the graph of a function from the xy-plane to the $x'y'$-plane. It is only a relation.

Definition 3.4 (Relation) *A relation on a space Z is a subset of $Z \times Z$. Viewing this in terms of iteration, an iterate of z under relation F is a point z' such that*

$(z, z') \in F$. *Notice that iterates of a point are not necessarily unique; nor do iterates necessarily exist.*

The relation in this example is a two-dimensional plane which is a subset of R^4 with second coordinate always equal to 0. A point $(x, y) \in R^2$ has no iterates unless $y = 0$. A point $(x, 0)$ has as iterates every point of the form (ax, y'), $y' \in R$. Thus the origin is still a "fixed" point under iteration. Since points on the x-axis have k^{th} iterates of the form $(a^k x, 0)$, which converge to the origin, the x-axis is in (and in fact equal to) the stable manifold. Likewise, every point on the y-axis is an iterate of the origin. Thus the y-axis is contained in (and in fact equal to) the unstable manifold.

We can also use the technique in Examples 3.1 and 3.2 to see this; although there is no longer a map, the limit of b increasing without bound corresponds to $b = \infty$; i.e. $\frac{1}{b} = 0$. Thus although our example is no longer a map, it is the graph of a function in skewed coordinates:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} ax \\ 0 \end{pmatrix}. \tag{3.10}$$

In this case, as in Examples 3.1 and 3.2, the limit of the iterates as expressed in skewed coordinates exists and is equal to the zero function. Again the projections of the graph of this zero function are the stable and unstable manifolds.

Example 3.5 Here is a contrived quadratic example to illustrate the same idea in a nonlinear case. Note that the map f on R^2 has a hyperbolic fixed point (0,0):

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax \\ b(y + cx^2) \end{pmatrix}, \qquad \text{for } 0 < a < 1 < b. \tag{3.11}$$

Since the axes are again the stable and unstable directions, we choose the axes for the skewed coordinate directions as before. The map represented in the skewed coordinate system gives the following function ϕ_1:

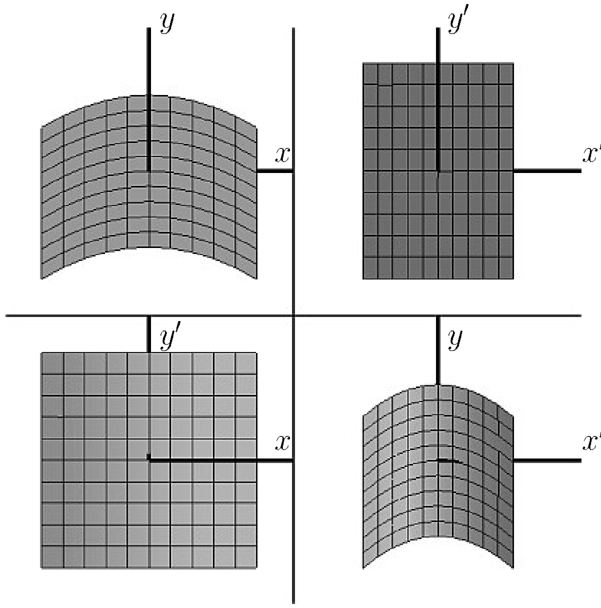

Figure 3.1: Projections of the graph of ϕ_1 resulting from the map in Example 3.5. Domain and range $[-.3, .3] \times [-.3, .3]$, $a = .7$, $b = 1.43$, and $c = 1$.

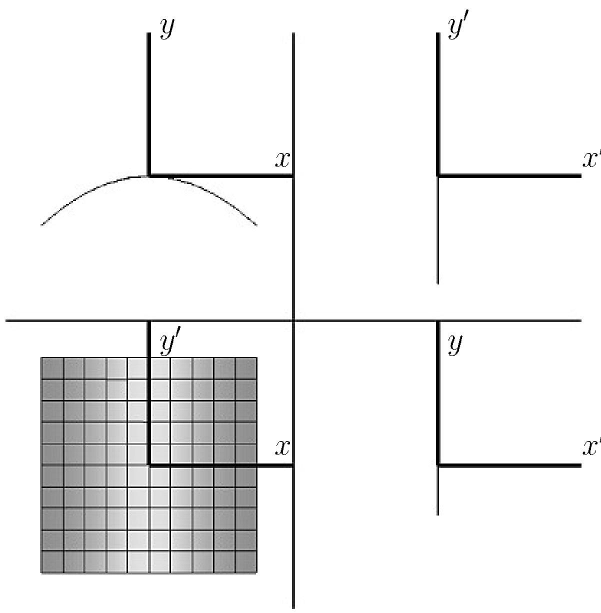

Figure 3.2: Same projections as in Figure 3.1, this time of ϕ_{20}, the skewed function of f^{20}. See also Figure 3.3.

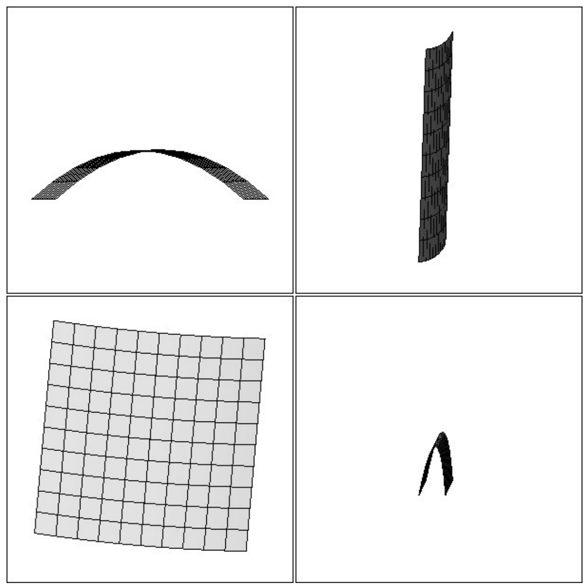

Figure 3.3: The graph of ϕ_{20} shown in Figure 3.2 after it has been rotated slightly in R^4. Same projections as before. This figure illustrates that although three of the four projections in the Figure 3.1 appear to be curves, the graph is actually a surface in R^4.

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} ax \\ \frac{1}{b}y' - cx^2 \end{pmatrix}. \tag{3.12}$$

Figure 3.1 shows the graph of ϕ_1 with domain $[-.3, .3] \times [-.3, .3]$. By the fact that ϕ_1 is a contraction, this figure is the same as the graph of f with both domain and range restricted to $[-.3, .3] \times [-.3, .3]$. Since the graph of a map from R^2 to R^2 is in R^4, the figure consists of projections of the graph to coordinate planes. The projections have the following relationship to the maps f and ϕ_1: f maps the region in the xy- plane to the region in the $x'y'$- plane. ϕ_1 maps the region in the xy'- plane to the region in the $x'y$- plane.

The k^{th} iterate f^k is:

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} a^k x \\ b^k(y + c(\frac{1-\mu^k}{1-\mu})x^2) \end{pmatrix}, \qquad \text{where } \mu = \frac{a^2}{b}. \tag{3.13}$$

Represented in skewed coordinates, it gives the following function ϕ_k:

$$\begin{pmatrix} x_k \\ y \end{pmatrix} = \begin{pmatrix} a^k x \\ \frac{1}{b^k}y_k - c(\frac{1-\mu^k}{1-\mu})x^2 \end{pmatrix}, \qquad \text{where } \mu = \frac{a^2}{b}. \tag{3.14}$$

Figure 3.2 shows the graph of ϕ_{20} for the same domain and constants as in Figure 3.1. Again, f^{20} maps the region in the xy- plane to the region in the $x'y'$- plane; ϕ_{20} maps the region in the xy'- plane to the region in the $x'y$- plane.

The limit $\lim_{k\to\infty} \phi_k$ exists. It is given by:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{c}{1-\frac{a^2}{b}}x^2 \end{pmatrix}. \tag{3.15}$$

As in the previous examples, the projections of the limit map to the xy- and $x'y'$- planes are respectively the local stable and unstable manifolds for f.

Since the convergence to the limit function is exponentially fast, the graph of ϕ_{20} in Figure 3.2 is visually indistinguishable from the graph of $\lim_{k\to\infty} \phi_k$. This is why three of the projections appear to be curves. However, the graphs of both

ϕ_{20} and the limit function are two-dimensional surfaces in R^4. To emphasize this point, Figure 3.3 shows projections of the same surface after it has been rotated in R^4[13].

The result from the above examples generalizes to a certain class of relations. In Section 3.3 we give basic definitions for the dynamics of relations and state the stable manifold theorem in this general setting. In Section 3.4, we outline the proof of the stable manifold theorem. Finally, in Section 3.5 we give the full details of the proof.

3.3 Basic definitions

In the previous section, relations on Z were defined as subsets of $Z \times Z$ and were viewed in terms of iteration. Here are some definitions in this context. We denote z having an iterate z' under relation f by $z \overset{f}{\mapsto} z'$.

Definition 3.6 (Fixed point) *Given a relation f on set Z, $z \in Z$ is a fixed point of f if $(z, z) \in f$.*

Definition 3.7 (Composition for relations) *Given relations g and h on set Z, $h \circ g$ is the relation given by*

$$\{(z, z'') : \exists\ z' \in Z, (z, z') \in g \text{ and } (z', z'') \in h\} \tag{3.16}$$

Notation: If I is an interval of integers and $z_k \in Z$ for all $k \in I$ is a sequence of points in Z, then we denote

$$\{z_k\}_{k \in I} = \left\{ \begin{array}{c} (z_i, z_{i+1}, ..., z_j), \text{ if } I = [i, j] \\ (..., z_{j-1}, z_j), \text{ if } I = (-\infty, j] \\ (z_i, z_{i+1}, ...), \text{ if } I = [i, \infty) \end{array} \right\} \tag{3.17}$$

Definition 3.8 (Orbits for relations) *Given relation f on space Z, an orbit through z is a sequence $\{z_k\}_{k \in I}$ such that $z = z_i$ for some $i \in I$, and $(z_k, z_{k+1}) \in f$ whenever $k, k+1 \in I$. If $I = [i, \infty)$ then $\{z_k\}$ is called an infinite forward orbit. If $I = (-\infty, i]$ then $\{z_k\}$ is called an infinite backward orbit.*

Definition 3.9 (Stable and unstable manifolds) *For a relation f on metric space Z with fixed point z_o, the stable and unstable manifolds $W^s(z_o)$ and $W^u(z_o)$ are defined by:*

$W^s(z_o) = \{z \in Z :$ there exists an infinite forward orbit $\{z_k\}$ through z such that $z_k \to z_o$ as $k \to \infty\}$.

$W^u(z_o) = \{z \in Z :$ there exists an infinite backward orbit $\{z_k\}$ through z such that $z_k \to z_o$ as $k \to -\infty\}$.

Definition 3.10 (C^r relations) *If f is a relation on a smooth manifold Z, then f is C^r when it is a C^r embedded submanifold of $Z \times Z$.*

Definition 3.11 (Linear relations) *If f is a relation on a vector space Z, then f is a linear relation if it is a linear subspace of $Z \times Z$.*

Definition 3.12 (Hyperbolic linear relations) *If f is an n-dimensional linear relation on an n-dimensional vector space Z, then f is hyperbolic when there is a splitting $Z = E^s \times E^u$ such that under this splitting, f is of the form*

$$\left\{ \begin{pmatrix} x \\ by' \\ ax \\ y' \end{pmatrix} : x \in E^s, y' \in E^u \right\}, \tag{3.18}$$

where a and b are matrices, and $|a|, |b| < 1$.

Note that the graph of any hyperbolic linear map is a hyperbolic linear relation. See Example 3.1 for the case of a saddle in R^2.

Definition 3.13 (C^r hyperbolic relations) *A C^r relation f on a smooth manifold Z has a hyperbolic fixed point z_o when $T_{(z_o,z_o)}f$, its tangent plane at (z_o, z_o), is a hyperbolic linear relation on $T_{(z_o,z_o)}Z \times Z$.*

Note that the graph of a map with hyperbolic fixed point z_o is a relation which has hyperbolic fixed point z_o.

We now state the main theorem of the chapter.

Theorem 3.14 (Stable manifold theorem for relations) *If f is a C^r relation on R^n, and f has hyperbolic fixed point z_o, then near z_o, $W^s(z_o)$ and $W^u(z_o)$ are graphs of C^r functions.*

3.4 Outline of the proof of the main theorem

The following definitions and lemmas outline the proof of the main theorem. The proofs of the lemmas are in the next section.

First note that for a relation f on R^n with a C^r hyperbolic fixed point z_o, f is locally the graph of a function $\underline{f}$. More precisely, for any μ and $k \leq r$, there is a neighborhood of z_o such that for some splitting $R^n = E^s \times E^u$ on this neighborhood, f is the graph of function $\underline{f}$, which is of the following form:

$$\underline{f}\begin{pmatrix} x \\ y' \end{pmatrix} = \begin{pmatrix} ax + \underline{g}_1(x, y') \\ by' + \underline{g}_2(x, y') \end{pmatrix} \tag{3.19}$$

where $x \in E^s$, $y' \in E^u$, a and b matrices, $|a|$, $|b| < 1$, and $\underline{g}_1$ and $\underline{g}_2$ functions which have all derivatives of order $\leq k$ Lipschitz with Lipschitz constant μ.

Motivated by this local expression of a hyperbolic relation as the graph of a function, we consider some definitions for relations on Euclidean space Z which are graphs of functions with certain properties for some coordinate system. We call these functions "associated" functions and call the coordinates "skewed" coordinates, represented by X and Y, where $Z = X \times Y$, and X and Y are Euclidean. Note that not every relation is the graph of such an associated function; these definitions are specifically intended for working with relations with hyperbolic fixed points. Also notice that the skewed coordinate system is not unique in any of the definitions below. However, once we choose a coordinate system, if there is an associated function in the coordinate system, then it is unique.

Notation: In the rest of this chapter, a relation is represented by a letter, and an associated function for this relation by the same letter underlined.

Definition 3.15 (Lipschitz relations) *A relation f is Lipschitz of order λ, or $f \in Lip_\lambda$, when there is an associated function $\underline{f} \in Lip_\lambda$ such that $(x, y) \overset{f}{\mapsto} (x', y') \Leftrightarrow \underline{f}(x, y') = (x', y)$.*

Lemma 3.16 *Suppose a relation $f \in Lip_\lambda$, $\lambda < 1$ has an associated Lipschitz function $\underline{f}$ as described in the above definition. Then the relation f is C^r exactly when the associated function $\underline{f}$ is C^r.*

The proof of the above lemma follows from the implicit function theorem. The result is tacitly assumed in the following lemma, which states that the composition of two Lipschitz and C^r relations gives another Lipschitz and C^r relation.

Lemma 3.17 *Let $\alpha < 1$ and $r \geq 0$. If g, Γ are relations in Lip_α and C^r on $Z = X \times Y$, with associated functions in the same skewed coordinates, then $\Gamma \circ g \in Lip_\alpha$ and C^r as well.*

Given relation f, for a relation ϕ, define G by $G(\phi) = f \circ \phi \circ f$. The following lemma says that for f with a hyperbolic fixed point and certain ϕ, G is a contraction.

Lemma 3.18 *Let f satisfy the hypotheses of theorem 3.14 and $\alpha < 1$. For suitably small neighborhood of the fixed point, assume ϕ is Lip_α with associated function in the same skewed coordinates as f. Note that $\{\phi\}$ lies in the Banach space of Lip_α relations in a fixed skewed coordinate system with the norm being the sup norm on the associated functions. Then G is a contraction in the sup norm on the associated functions.*

Since G is a contraction in the space of Lip_α relations, G has a unique fixed point which is also in the space of Lip_α relations, and any such relation converges to this fixed point. In fact we can choose a neighborhood Ω such that f is an appropriate Lipschitz relation in the domain of G. Thus on this neighborhood, the fixed point is equal to $\lim_{k\to\infty} f^k$. Call this fixed point relation h and its associated function $\underline{h}$. The above lemma guarantees that h is Lipschitz on Ω. In fact, h is also C^r on Ω, as is restated below.

Lemma 3.19 *Assume $f \in C^r$ satisfying the hypotheses in theorem 3.14, and for Ω a neighborhood of z_o such that on Ω the fixed point relation is Lipschitz and equal to $\lim_{k\to\infty} f^k$. Then h is C^r on Ω.*

Definition 3.20 (ω-limit relation) *Given the relation f on compact metric space Z,*

$$f^\omega = \cap_{n\geq 1}\overline{\cup_{k\geq n} f^k},$$

where f^k is the composition of k copies of f.

The following lemma states that the relation h defined above is equal to the ω-limit relation:

Lemma 3.21 $f^\omega = h$.

The next two lemmas state that the ω-limit relation is locally the cross product of the stable and unstable manifolds.

Lemma 3.22 *For a relation f satisfying the hypotheses of theorem 3.14, there is a neighborhood of the fixed point such that if $u \in W^s(z_o)$ and $v \in W^u(z_o)$, then (u, z_o) and (z_o, v) are contained in f^ω.*

In fact, a stronger statement holds; the following lemma states that f^ω relates every point in $W^s(z_o)$ to every point of $W^u(z_o)$.

Lemma 3.23 *For a relation f satisfying the hypotheses of theorem 3.14, there is a neighborhood of the fixed point such that $u \in W^s(z_o)$ and $v \in W^u(z_o) \Leftrightarrow (u, v) \in f^\omega$.*

We now use the lemmas stated in this section to prove the stable manifold theorem for relations.

Proof of theorem 3.14: By lemma 3.23, the stable and unstable manifolds are projections of f^ω. Precisely, $f^\omega = W^s(z_o) \times W^u(z_o)$. By lemma 3.22, this set equals $\{u : (u, z_o) \in f^\omega\} \times \{v : (z_o, v) \in f^\omega\}$. By lemma 3.21, $h = f^\omega$; by lemmas 3.19 and 3.16, h has an associated C^r function $\underline{h}$. In terms of the splitting, denote $z_o = (x_o, y_o)$. $W^s \times W^u = \{(x, y) : \underline{h}(x, y_o) = (x_o, y)\} \times \{(z, w) : \underline{h}(x_o, w) = (z, y_o)\}$. Thus both W^s and W^u are locally the graphs of C^r functions. □

3.5 Proofs of lemmas

Proof of lemma 3.17: The proof is an application of the C^r and Lipschitz implicit function theorems. Since it is less common than the C^r implicit function theorem, we state the Lipschitz version here.

Theorem 3.24 (Lipschitz implicit function theorem) *If X and Y are metric spaces, and $\underline{F} : X \times Y \to X$ is a continuous mapping $\underline{F} \in Lip_\lambda$, $\lambda < 1$, then there exists function $\underline{g} : Y \to X$, $\underline{g} \in Lip_\lambda$ such that*

$$\underline{F}(x, y) = x \Leftrightarrow x = \underline{g}(y).$$

Proceeding with the proof of lemma 3.17, we need to show that if $g, \Gamma \in$Lip$_\alpha$ and C^r, then there exists a Lip$_\alpha$ and C^r function $\underline{\Gamma \circ g}$ such that $(x, y, x'', y'') \in \Gamma \circ g$ exactly when $\underline{\Gamma \circ g}(x, y'') = (x'', y)$. Define a function $\underline{F} : Z \times Z \times Z \to Z \times Z$ by

$$\underline{F}((x'', y'), (x', y), (x, y'')) = (\underline{\Gamma}(x', y''), \underline{g}(x, y')). \tag{3.20}$$

Since $\underline{F}$ is Lip$_\alpha$ and has no unit norm eigenvalues, by the implicit function theorem, there exists a Lip$_\alpha$ and C^r function $\underline{m} : Z \to Z \times Z$ such that $F(x'', y', x', y, x, y'') = (x'', y', x', y)$ exactly when $\underline{m}(x, y'') = (x'', y', x', y)$. Thus $(\underline{m}_1, \underline{m}_4) = \underline{\Gamma \circ g}$. □

Proof of lemma 3.18: This proof is a series of estimates. The key to the estimates is that f and the domain of G are Lipschitz.

Assume that f is as in the theorem, and we have picked a neighborhood and splitting so that equation 3.19 holds and $\underline{g}_1, \underline{g}_2$ are Lip$_\mu$ functions. Let $\lambda = \max(|a|, |b|) + \mu$. Assume we have chosen a small enough neighborhood that $\lambda + \alpha\mu < 1$.

(Note that $f \in$ Lip$_\lambda$.)

For a relation ψ with associated function $\underline{\psi}$, let $\|.\|$ denote the sup norm, and let $\underline{\psi} = (\psi_1, \psi_2)$ be the components of the associated function.

We want to show that for any relations $\phi, \psi \in$ Lip$_\alpha$, there is some uniform constant $\theta < 1$ such that $\|\underline{G(\psi)} - \underline{G(\phi)}\| < \theta\|\underline{\psi} - \underline{\phi}\|$. This is equivalent to showing that $\sup_{x=\xi, y'''=\eta'''} |(x, y, x''', y''') - (\xi, \eta, \xi''', \eta''')| < \theta\|\underline{\psi} - \underline{\phi}\|$, where $(x, y, x''', y''') \in G(\phi)$, and $(\xi, \eta, \xi''', \eta''') \in G(\psi)$.

If $(x, y, x''', y''') \in G(\phi)$, and $(\xi, \eta, \xi''', \eta''') \in G(\psi)$, then there exist x',y',x'',y'', ξ', η', ξ'', and η'' such that

$$\begin{aligned} (x, y) &\overset{f}{\mapsto} (x', y') \overset{\phi}{\mapsto} (x'', y'') \overset{f}{\mapsto} (x''', y''') \\ (\xi, \eta) &\overset{f}{\mapsto} (\xi', \eta') \overset{\psi}{\mapsto} (\xi'', \eta'') \overset{f}{\mapsto} (\xi''', \eta''') \end{aligned} \tag{3.21}$$

The following inequalities hold:

$$
\begin{aligned}
|x''' - \xi'''| &= |ax'' + \underline{g}_1(x'', y''') - a\xi'' - \underline{g}_1(\xi'', \eta''')| &(3.22)\\
&\leq \lambda|x'' - \xi''|, \text{ since } y''' = \eta''' \\
\text{and } |x'' - \xi''| &= |\phi_1(x', y'') - \psi_1(\xi', \eta'')| &(3.23)\\
&\leq \alpha \max(|x' - \xi'|,\ |y'' - \eta''|) + \|\underline{\phi} - \underline{\psi}\|, \text{ since } \psi, \phi \in \mathrm{Lip}_\alpha
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|x' - \xi'| &\leq \mu|y' - \eta'| &(3.24)\\
|y'' - \eta''| &\leq \mu|x'' - \xi''| &(3.25)\\
|y' - \eta'| &\leq \alpha \max(|x' - \xi'|,\ |y'' - \eta''|) + \|\underline{\phi} - \underline{\psi}\| &(3.26)\\
|y - \eta| &\leq \lambda|y' - \eta'| &(3.27)
\end{aligned}
$$

If we let $\Delta = \max(|x'' - \xi''|, |y' - \eta'|)$, then from the above equations, we have

$$
\begin{aligned}
\Delta &\leq \alpha\mu\Delta + \|\underline{\phi} - \underline{\psi}\|, \text{ so} &(3.28)\\
\Delta &\leq \frac{1}{1-\alpha\mu}\|\underline{\phi} - \underline{\psi}\|
\end{aligned}
$$

Thus for $\theta = \frac{\lambda}{1-\alpha\mu} < 1$, which is guaranteed by our original assumption, $\sup_{x=\xi, y'''=\eta'''} |(x, y, x''', y''') - (\xi, \eta, \xi''', \eta''')| < \theta\|\underline{\psi} - \underline{\phi}\|$. □

Proof of lemma 3.19: To show that $h \in C^r$ when $f \in C^r$, we first show that there is a neighborhood of the fixed point of f such that the limit relation of f restricted to this neighborhood is C^r. To do this, we use the fiber contraction theorem [12] to show that the map G is a C^1 contraction when f is C^1. G is locally a C^r contraction when $f \in C^r$ by an induction argument. In order to show that h is a C^r relation on the original neighborhood, the relationship between h and the limit relation on a smaller neighborhood bears further comment. To this end, we prove that h is equal to the limit relation on the smaller neighborhood composed with finitely many C^r subsets of f. Therefore h is C^r on the entire original neighborhood. We use the following definitions and lemmas; the central proof follows their statements and proofs.

The following is a definition of a derivative relation of a smooth relation.

Definition 3.25 (Tangent relation) *Given a smooth relation Γ on R^p, the tangent relation $T\Gamma$ on R^{2p} is the tangent bundle of Γ.*

If a relation has an associated function, then its tangent relation has an associated function, as described in the following lemma.

Lemma 3.26 *For a smooth relation Γ on $R^p = X \times Y$ with associated function $\underline{\Gamma}$, $T\Gamma$ is the graph of $(\underline{\Gamma}, D\underline{\Gamma})$. In other words, $(x, y, x', y', \xi, \eta, \xi', \eta') \in T\Gamma$ exactly when $(x, y, x', y') \in \Gamma$ and $D\underline{\Gamma}(x, y')(\xi, \eta') = (\xi', \eta)$.*

Proof This is due to the fact that a graph of a smooth function has tangent bundle equal to the graph of the derivative of the function. □

Lemma 3.27 (Derivatives and composition) *Assume that Γ and g are smooth relations with associated functions, and $(x, y, x'', y'') \in \Gamma \circ g$. Then locally there exist x' and y' such that the graph of $D(\underline{\Gamma \circ g})_{(x,y'')}$ is equal to graph$(D\underline{\Gamma}_{(x',y'')}) \circ$ graph$(D\underline{g}_{(x,y')})$. In terms of tangent relations, locally $T\Gamma \circ Tg = T(\Gamma \circ g)$.*

Proof of lemma 3.27: In the proof of lemma 3.17, we showed that locally there are unique x' and y' which are functions of (x, y'') such that

$$(x, y) \overset{g}{\mapsto} (x', y') \overset{\Gamma}{\mapsto} (x'', y''). \tag{3.29}$$

We know that $y' = \underline{\Gamma}_2(\underline{g}_1(x, y'), y'')$. For the coordinate system $R^n = E^s \times E^u$, write the derivative matrices in the form $D\underline{g} = \begin{pmatrix} D_1\underline{g}_1 & D_2\underline{g}_1 \\ D_1\underline{g}_2 & D_2\underline{g}_2 \end{pmatrix}$. Implicit differentiation gives

$$Dy' = (1 - D_1\underline{\Gamma}_2 D_2 \underline{g}_1)^{-1}(D_1\underline{\Gamma}_2 D_1 \underline{g}_1, D_2\underline{\Gamma}_2), \tag{3.30}$$

where all derivatives are evaluated at (x, y, x', y', x'', y''). It is now possible to write the derivative of $\Gamma \circ g$ explicitly. Comparing this derivative to the function associated with graph $D\underline{\Gamma}_{(x',y'')}\circ$ graph $D\underline{g}_{(x,y')}$ shows that they are equal. □

Lemma 3.28 *Let $\alpha < 1$, g and Γ both be Lip_α C^r relations on the compact set V; assume that in the coordinates $V = V_1 \times V_2$, there is a contraction $\underline{g}$ associated with g. Also assume that for $U_1 \subset V_1$ and $U_2 \subset V_2$, $\underline{g} : U_1 \times V_2 \to V$ and $\underline{g} : V_1 \times U_2 \to V$. Let Γ be a relation on V. Then $g \circ \Gamma \circ g$ is Lip_α and C^r on $U_1 \times U_2$.*

Proof of lemma 3.28: Define the function $\underline{F} : V \times V \times V \times U \to V \times V \times V$ as $\underline{F}((x'', y), (x', y''), (x''', y'), (x, y''')) = (\underline{g}(x'', y'''), \underline{\Gamma}(x', y''), \underline{g}(x, y'))$. Proceed using the implicit function theorem as in the proof of lemma 3.17. □

The final lemma is the fiber contraction theorem due to Hirsch and Pugh. Its proof can be found in [12].

Lemma 3.29 (Fiber contractions) *Let Ψ be a map on a space X with attractive fixed point p. For each $x \in X$, let Υ_x be a map on metric space Y such that $\Theta(x, y) = (\Psi(x), \Upsilon_x(y))$ is continuous on $X \times Y$. For fixed $\lambda < 1$ and each x, let each $\Upsilon_x \in Lip_\lambda$. Then there is an attracting fixed point (p, q) for Θ.*

Finally, the following definition makes the notation more convenient:

Definition 3.30 ((C^r,ϵ) and (Lip^r,ϵ) Small Relations) *A relation is (C^r, ϵ) small if there is some associated function which is (C^r, ϵ) small; in other words, there is an associated function which is C^r, and all its derivatives of order $< r$ are Lip_ϵ. A relation is (Lip^r,ϵ) small if it is (C^r, ϵ) small and the r^{th} derivative of the associated function is Lip_ϵ.*

Using these lemmas, the proof of lemma 3.19 proceeds as follows: Let f be a C^r relation on R^n with hyperbolic fixed point at z_o, as in theorem 3.14. Let Ω be a neighborhood of the fixed point such that $\lim_{j\to\infty} f^j = h$, as described in the discussion after the statement of lemma 3.18. Let μ and λ be as in the proof of lemma 3.18, and let $z_o = (x_o, y_o)$ in terms of the splitting.

First we show G is a C^1 contraction when f is C^1 ($r = 1$). Since Tf is not necessarily Lipschitz, we cannot just apply lemma 3.18 on the tangent bundle. However, as in the diffeomorphism case, we can still prove the result using the fiber contraction theorem.

Let ϕ be a Lip_λ relation on R^{2n} such that $\phi = (\rho, L)$, ρ a relation on R^n, and $\underline{L}(x, y', .)$ linear. Consider the map $TG : \phi \to Tf \circ \phi \circ Tf$. We verify the conditions for the fiber contraction theorem for TG; $\pi_1 TG$ has attractive fixed point h. Near (x_o, y_o), $D\underline{f}(x, y')$ is close to $D\underline{f}(x_o, y_o)$ in linear norm. Thus we can use estimates similar to those in the proof of lemma 3.18 on $\pi_2 TG$ on a neighborhood of $(x_o, y_o, 0, 0)$. On such a neighborhood, for fixed ρ and varying L, $\pi_2 TG$ is Lip_λ in the sup norm. By the fiber contraction theorem, TG is a contraction in the sup norm on relations ϕ above. Thus TG is a contraction when $\phi = T\rho$, where ρ is a (Lip^1,μ) small relation on R^n. Therefore G is a C^1 contraction on (Lip^1,μ) small relations.

For the case $r > 1$, proceed by induction. Assume that for all relations $g \in C^{r-1}$ on with hyperbolic fixed point on R^p, and for ρ (Lip^{r-1},μ) small, that $\rho \to g \circ \rho \circ g$ is a C^{r-1} contraction. Choose $\tilde{g}$ C^r, hyperbolic. By our assumption, $T\tilde{\rho} \to T\tilde{g} \circ T\tilde{\rho} \circ T\tilde{g}$ is a C^{r-1} contraction when $\tilde{\rho}$ (Lip^r,μ) small relations. Therefore $\tilde{\rho} \to \tilde{g} \circ \tilde{\rho} \circ \tilde{g}$ is a C^r contraction.

We have so far shown that for $f \in C^r$, there is an ϵ such that on a ball of radius ϵ of the fixed point, G is a contraction in the C^r sup norm on (Lip^r, μ) small relations. The limit relation on this small ball is thus C^r. We now use the smoothness of the limit relation on the small balls to show that the relation h is C^r on all of Ω.

Choose r. Denote the ball of radius ϵ of the fixed point by $B^1_\epsilon \times B^2_\epsilon$. Let Γ be the C^r fixed point of G restricted to this ϵ ball. Note that $\Gamma \subset h$, since it can be described as the limit of iteration of the relation $f|_{B^1_\epsilon \times B^2_\epsilon}$.

For λ described in the proof of lemma 3.18, if $|(x,y)-(x_o,y_o)| < \frac{\epsilon}{\lambda}$, then $|\underline{f}(x,y)-(x_o,y_o)| < \epsilon$. Thus $\underline{f} : B^1_{\frac{\epsilon}{\lambda}} \times B^2_\epsilon \to B^1_\epsilon \times B^2_\epsilon$ and $\underline{f} : B^1_\epsilon \times B^2_{\frac{\epsilon}{\lambda}} \to B^1_\epsilon \times B^2_\epsilon$. Thus by lemma 3.28, $f \circ \Gamma \circ f$ is Lip_λ and C^r when restricted to the set $B^1_{\frac{\epsilon}{\lambda}} \times B^2_{\frac{\epsilon}{\lambda}}$.

This new relation is also a subset of h since if $f \subset f'$ and $\Gamma \subset \Gamma'$, then $f \circ \Gamma \subset f' \circ \Gamma'$. We know that $\Gamma \subset h$. Thus $f \circ \Gamma \circ f \subset f \circ h \circ f = h$.

Now iterate this process of composing with f and restricting to a neighborhood; eventually we have a $\mathrm{Lip}_\lambda, C^r$ relation on Ω. Since this relation is contained in h and both are associated with functions on Ω, the relations must be equal. Thus h is C^r on Ω. □

Proof of lemma 3.21: Assume that we have a neighborhood and splitting for f as in equation 3.19. We show that there is a sequence in f^k converging to a limit point (x,y,z,w) exactly when there is a sequence k_i such that $\lim_{i\to\infty} \underline{f^{k_i}}(x,w) = (z,y)$.

First we show that $h \subset f^\omega$; from the definition,

$$f^\omega = \left\{ u : u_{k_i} \to u \text{ for some } u_{k_i} \in f^{k_i} \right\}. \tag{3.31}$$

By lemma 3.18, $f \mapsto f \circ f \circ f \mapsto f^5 \mapsto \ldots$ maps to h in the sup norm on the associated functions. Thus for all (x,w) and odd k, $\underline{f^k}(x,w)$ has a limit, and the limit is equal to $\underline{h}(x,w)$. Thus $u_k = (x, \underline{f^k}_2(x,w), \underline{f^k}_1(x,w), w)$ shows that $h \subset f^\omega$.

Conversely, to show that $f^\omega \subset h$, suppose $\eta = (x,y,z,w) \in f^\omega$, and $u_{k_i} = (x_{k_i}, y_{k_i}, z_{k_i}, w_{k_i})$ is the sequence in f^{k_i} guaranteed by equation 3.31 such that $|u_{k_i} - \eta| \to 0$. Define $v_{k_i} = (x, \underline{f}^{k_i}_2(x,w), \underline{f}^{k_i}_1(x,w), w)$. Then

$$| \eta - v_{k_i} | \leq | \eta - u_{k_i} | + | u_{k_i} - v_{k_i} | \, . \tag{3.32}$$

The first term on the right goes to zero by construction. In addition, since f is in Lip_λ, the second term is less than or equal to the first term. Therefore it goes to zero as well. Thus $\eta \in h$. □

Proof of lemma 3.22 follows from lemma 3.23. □

Proof of lemma 3.23: Assume that we have a neighborhood and splitting described in equation 3.19 and that in terms of the splitting, the fixed point is denoted by (x_o, y_o). Assume $(x, y) \in W^s(x_o, y_o)$ and $(z, w) \in W^u(x_o, y_o)$. In the proof that follows, we look at the forward k-iterates of a neighborhood of (x, y) and the backward k-iterates of a neighborhood of (z, w). For large k, near the fixed point, a portion of the forward iterates form a Lipschitz "vertical" curve, and a portion of the backward iterates form a Lipschitz "horizontal" curve. The two curves are near each other, and thus intersect, implying the existence of a point near (x, y) with a $2k$-iterate near (z, w). More precisely, we use this idea to show that for any ϵ, K there exists $k > K$ and a point $(s, t, u, v) \in f^k$ such that $\text{dist}((x, y, z, w), (s, t, u, v)) < \epsilon$ and thus $(x, y, z, w) \in f^\omega$.

Let ϵ be given. We know that there exist sequences (x_k, y_k), and (z_k, w_k) both converging to the fixed point, $(x, y, x_k, y_k) \in f^k$ and $(z_k, w_k, z, w) \in f^k$. For a small δ, let k be large enough that distance from (x_k, y_k) to (z_k, w_k) is less than δ.

Now look at an ϵ ball of y_k in Y. For the point (x, η'), where η' is in the ϵ ball, we have the point $(x, \eta, \xi', \eta') \in f^k$. The set of points (ξ', η') form the graph of a Lipschitz function from Y to X near (x_k, y_k), each point of which is related to a point near (x, y) by f^k. Similarly, there is a graph of a Lipschitz function from X to Y near (z_k, w_k), and a point near (z, w) is related to each of the points in this graph. But if δ is small enough, these Lipschitz graphs must intersect.

Thus there is a point $(s,t,u,v) \in f^{2k}$ within ϵ of (x,y,z,w). We conclude that $(x,y,z,w) \in f^\omega$.

Conversely, assume $(x,y,z,w) \in f^\omega$. Therefore for any $k > 0$, there are points near (x,y) with k forward iterates. Using compactness, we show that (x,y) has an infinite forward orbit. Using the fact that $f \in Lip_\lambda$, we show that the forward orbit must converge to the fixed point, and thus $(x,y) \in W^s(x_o,y_o)$. Likewise, $(z,w) \in W^u(x_o,y_o)$.

Let $B_\epsilon(x,y)$ be the closed ϵ ball of (x,y), and define the set

$$S_\epsilon^k(x,y) = \left\{(\xi,\eta) \in B_\epsilon(x,y) : (\xi,\eta) \text{ has a } k^{th} \text{ iterate}\right\}.$$

$S_\epsilon^k(x,y)$ is nonempty, by the assumption on (x,y). It is compact, since f is closed, which implies f^k closed [18] and thus compact. Thus $\cap_\epsilon S_\epsilon^k(x,y)$ is nonempty, since it is the intersection of non-empty nested compact sets. It is equal to $\{(x,y)\}$, since this is the only point it could contain. Therefore (x,y) has a k^{th} iterate (x_k,y_k) for every k. Thus there exists an infinite forward orbit starting at (x,y). By compactness, there exists a limit point (z',w'). Thus $(x_j,y_j,z',w') \in f^\omega$ for the same z',w' for all j.

Since $(x_j,y_j,z',w') \in f^\omega$ for all j, if (x_k,y_k) and (x_j,y_j) are in this forward orbit, then $|y_j - y_k| < \lambda|x_j - x_k|$.

Since f^k is Lip_λ for every k,$|x_{k+1} - x_k| < \lambda \max(|x_{k-1} - x_k|, |y_{k+1} - y_k|) < \lambda \max(|x_{k-1} - x_k|, \lambda|x_{k+1} - x_k|)$. Thus $|x_{k+1} - x_k| < \lambda|x_{k-1} - x_k|$. This Cauchy sequence implies that x_k converges to a unique z_o. Likewise, y_k converges to a unique w_o. This means that $(x_k,y_k,x_{k+1},y_{k+1}) \to (z_o,w_o,z_o,w_o)$. Since f is closed and $(x_k,y_k,x_{k+1},y_{k+1}) \in f$, $(z_o,w_o,z_o,w_o) \in f$ as well. The unique fixed point of f is (x_o,y_o). Therefore (x_k,y_k) converges to (x_o,y_o). Therefore $(x,y) \in W^s(x_o,y_o)$. Similarly, $(z,w) \in W^u(x_o,y_o)$. □

This completes the proof of the stable manifold theorem.

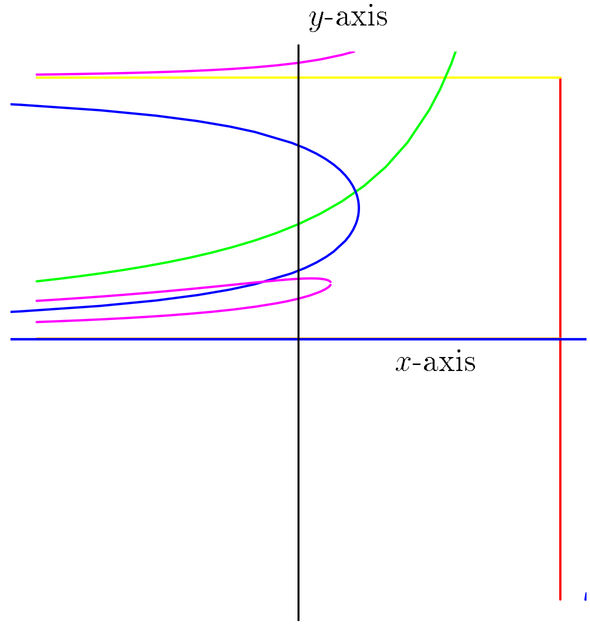

Figure 3.4: A portion of the global stable manifold of the origin in the delayed regulation map. This could never be a stable manifold for a fixed point of a diffeomorphism, since it has self-intersections.

3.6 Global manifolds

So far we have talked only about local results for stable and unstable manifolds of fixed points. In other words, for relation F, and Ω a neighborhood of hyperbolic fixed point p, we have looked only at $W^s_{loc} = W^s(p, F \cap \Omega)$. For diffeomorphisms, even if we don't restrict to looking at the relation $F \cap \Omega$, a global theorem follows from the local stable manifold theorem. Although the global manifolds may no longer be embedded submanifolds, they still have a smooth structure.

Definition 3.31 (Immersed manifolds) *A subspace S of an ambient space M is called an immersed submanifold if there is a smooth manifold N, and a one-to-one smooth function with derivative an isomorphism, mapping N to S.*

Theorem 3.32 *For a diffeomorphism, the global stable and unstable manifolds of a hyperbolic fixed point are immersed submanifolds.*

This theorem follows from:

$$W^s(p, f) = \cup_{k \geq 0} f^{-k}(W^s_{loc}(p))$$

$$W^u(p, f) = \cup_{k \geq 0} f^{k}(W^u_{loc}(p)).$$

Thus we can use the smoothness of the local manifolds, and the fact that f is a diffeomorphism, to get the smooth structure of the global manifolds. In contrast, a noninvertible map is not one-to-one, and thus cannot be used to show anything about structure of the global manifolds. In fact, generally the global manifolds for noninvertible maps fail to have any smooth structure.

The diffeomorphism theorem guarantees that global stable and unstable manifolds for diffeomorphisms are connected, locally smooth, of a fixed dimension, and with no self-intersections. These properties all can fail for noninvertible maps; the stable manifold may have infinitely many components, an image under a noninvertible map may have a cusp [7], it may collapse in dimension [11], or it may have self-intersections, as occurs in the global stable manifold to the origin in the delayed regulation map. See Figure 3.4.

In general, it does not seem that there is anything special to distinguish global stable and unstable manifolds of relations from arbitrary closed sets. The theorems we can expect to hold for noninvertible maps and relations are those that are local in nature.

Chapter 4

Hyperbolic Sets

This chapter contains a definition of hyperbolic sets for smooth relations. It then presents a proof of the shadowing lemma and stable manifold theorem for hyperbolic sets.

4.1 Background

For diffeomorphisms, the idea of hyperbolic fixed points generalizes to sets of points which are not necessarily fixed. Here is a precise definition.

Definition 4.1 (Hyperbolic sets for diffeomorphisms) *A compact invariant set K for a diffeomorphism f is a hyperbolic set if the following conditions hold:*

1. *There is a continuous splitting of the tangent space into stable and unstable subspaces,*
$$T_x R^n = E^s_x \times E^u_x, \;\; x \in K.$$
2. *The derivative is invariant for the splitting,*
$$Df(x) E^s_x = E^s_{f(x)},$$

$$Df(x)E^u_x = E^u_{f(x)}.$$

3. *The derivative is eventually contracting on the stable subspaces and eventually expanding on the unstable subspaces. That is, for $C > 0$, $\lambda < 1$,*

$$|Df^n(x)v| < C\lambda^n|v|, \quad v \in E^s_x,$$

$$|Df^n(x)v| > C^{-1}\lambda^{-n}|v|, \quad v \in E^u_x.$$

By a theorem due to Mather [24], there is a continuous metric such that $C = 1$, and thus the derivative contracts stable subspaces and expands unstable subspaces under just one iterate. This is called the *adapted metric*.

Hyperbolic sets for diffeomorphisms have been carefully studied, and there are many theorems describing the dynamics on such sets. Thus we would like to generalize the definition and corresponding theorems to noninvertible maps.

4.1.1 Why have a new definition?

For noninvertible maps, we cannot just apply the definition of hyperbolic sets as it stands. The requirement that stable and unstable subspaces map invariantly is too stringent. Trivially, if the derivative is singular, then we can only expect $Df(x)E^s_x \subset E^s_{f(x)}$. In addition, we cannot expect the unstable subspaces to map invariantly either, since there may be multiple points mapping to the same point. For example, in the delayed regulation model at $a > a^* = 2.2701$, there are transverse homoclinic orbits to the fixed point. The closure of such an orbit is a likely candidate for a hyperbolic set. However, for such a transverse homoclinic orbit, it is impossible to define invariant unstable directions at every point. This arises from the fact that points on the x-axis map directly to the fixed point, rather than converging to it; in the two homoclinic orbits, there exist points $q = (q_1, 0)$

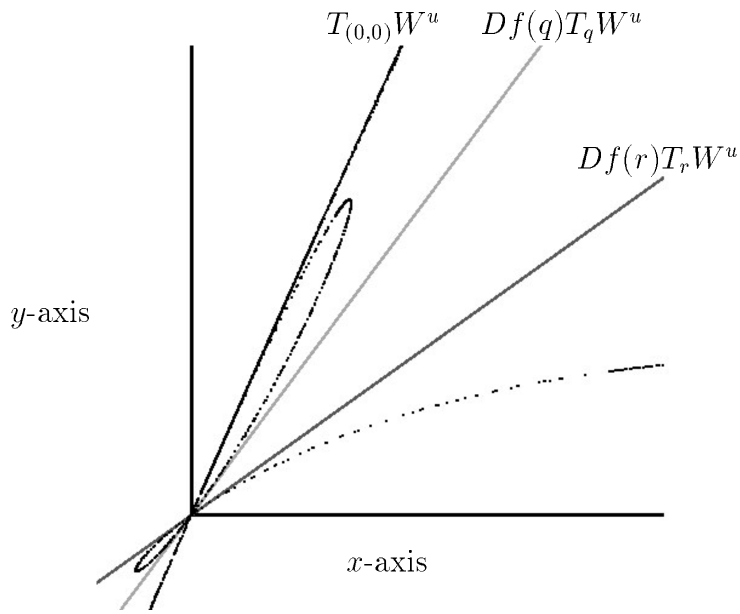

Figure 4.1: W^u near the origin for the delayed regulation model. The three tangents to W^u show that there is no invariantly mapping unstable subspace.

and $r = (q_2, 0)$ such that $f(q) = f(r) = (0, 0)$. Since the derivative

$$Df(x,y) = \begin{pmatrix} 0 & 1 \\ -ay & a(1-x) \end{pmatrix} \tag{4.1}$$

maps point q and r by $Df(q_i, 0)(\xi, 1) = (1, a(1-q_i))$, there is no vector in the tangent space of q or r which maps by the derivative to $(1, a)$, the unstable eigenspace for the fixed point. Thus the derivative cannot map invariantly on the unstable subspaces. This phenomenon was first described in work of Steinlein and Walther [27]. Figure 4.1 shows the global unstable manifold in a neighborhood of the origin. The three lines are candidates for $E^u_{(0,0)}$: the unstable eigenspace $(1, a)$ of the derivative at the origin, and $(1, a(1-q_1))$ and $(1, a(1-q_2))$, the images under the derivative of the tangent vector corresponding to the homoclinic points q and r on the x-axis.

Based on these ideas, Steinlein and Walther [26, 27] generalize the definition of hyperbolic sets to a definition for C^1 noninvertible maps on Banach spaces. For their definition, they assume that there exist stable subspaces mapping invariantly, and eventually contracting. They also assume that there exist unstable subspaces, not mapping invariantly, but with unstable projections eventually expanding. For this definition, they prove the shadowing lemma by making some adjustments to the traditional functional analytic techniques. My development of the theory of hyperbolic sets follows a very different approach. Using the ideas in the proof of the stable manifold theorem, I give a symmetric definition of hyperbolic sets, which also applies to smooth relations. From this definition, I prove the shadowing lemma using a proof, which though functional analytic, differs significantly from the traditional approach. I also give a proof of a version of the stable manifold theorem for hyperbolic sets.

4.2 Definitions

Definition 4.2 (Splitting for R^n) *A splitting for R^n is a pair of subspaces (E^s, E^u) satisfying*

$$R^n = E^s_x \times E^u_x.$$

Definition 4.3 (4-Splitting for $R^n \times R^n$) *A 4-splitting for $R^n \times R^n$ is a pair of splittings for R^n, (E^s_1, E^u_1) and (E^s_2, E^u_2), satisfying dim(E^s_1)=dim(E^s_2) and dim(E^u_1) = dim(E^u_2).*

Definition 4.4 (Hyperbolic Linear Relation with respect to a 4-splitting) *If A is an n-dimensional linear relation on R^n, then A is hyperbolic with respect to the 4-splitting $(E^s_1, E^u_1, E^s_2, E^u_2)$ if there is a linear contracting map $M : E^s_1 \times E^u_2 \to E^s_2 \times E^u_1$, $|M| < \lambda < 1$ such that A is the graph of M. Thus*

$$A = \left\{ (x, y, x', y') : \begin{pmatrix} x' \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y' \end{pmatrix} \right\}.$$

We say A has a contraction constant $|M|$ with respect to this 4-splitting.

Notation: For the rest of the thesis, a capital F represents a relation, whereas a lower case f represents a function.

Definition 4.5 (Compact hyperbolic sets for C^1 relations) *Let F be a C^1 relation on R^n. Let K be a compact set in R^n. K is a hyperbolic set for F when the following conditions hold:*

There is a continuous splitting of the tangent space of R^n over K. Denote it: $T_z R^n = E^s_z \times E^u_z$.

There is a constant $\lambda < 1$.

There is some metric on a neighborhood of K such that at every $(z, w) \in F \cap (K \times K)$, $T_{(z,w)}F$ is a hyperbolic linear relation with contraction constant $< \lambda$ with respect to the 4-splitting $(E^s_z, E^u_z, E^s_w, E^u_w)$.

The following lemma relating hyperbolicity and Lipschitz relations is what we actually use in the proofs of the theorems in the next section.

Lemma 4.6 (Hyperbolic implies Lipschitz) *If $(z, w) \in F \cap (K \times K)$, with F and K as above, then in a neighborhood of (z, w), F is the graph of a Lip_λ function $f : E^s_z \times E^u_w \to E^s_w \times E^u_z$.*

Proof By the Implicit Function Theorem. □

4.3 Cones

Although hyperbolic sets for diffeomorphisms always have invariant subspaces, it is often difficult to find them. To avoid this problem, hyperbolicity is often formulated in terms of an equivalent condition on stable and unstable cones. We show below that a hyperbolic set for a relation satisfies a generalization of the cone condition.

For diffeomorphisms, the cone formulation says that on some compact invariant set K, if the derivative is expanding on a cone of vectors roughly in the unstable direction and contracting on a cone of vectors roughly in the stable direction, then K is a hyperbolic set for the diffeomorphism [21]. The following makes this more precise.

Definition 4.7 (Stable and unstable cones) *Given $\alpha > 0$, and a splitting of the tangent space at each point $T_z R^n = E^s_z \times E^u_z$, then the stable and unstable α cones are defined by:*

$$C^s_z = \{(v_s, v_u) \in E^s_z \times E^u_z : |v_u| \leq \alpha |v_s|\}$$

$$C^u_z = \{(v_s, v_u) \in E^s_z \times E^u_z : |v_s| \leq \alpha |v_u|\}.$$

Definition 4.8 (Cone condition for diffeomorphisms) *If f is a diffeomorphism and K is a compact invariant set, then K satisfies the cone condition if there is a continuous metric and a continuous splitting of K such that $Df(z)C^s_z \subset C^s_{f(z)}$, $Df^{-1}(f(z))C^u_{f(z)} \subset C^u_z$, and Df is uniformly contracting on the stable cone and uniformly expanding on the unstable cone.*

Theorem 4.9 (Hyperbolic sets and cones for diffeomorphisms) *If f is a diffeomorphism and K is a compact invariant set, then K satisfies the cone condition if and only if K is a hyperbolic set.*

This theorem is due to Newhouse and Palis [21].

The new definition of hyperbolic sets for relations is similar to this cones formulation. Here is a natural generalization of the cone condition to relations, and a statement that hyperbolic sets for relations satisfy this cone condition.

Definition 4.10 (Cone condition for relations) *Let F be a smooth relation and K a compact set. Then F satisfies the cone condition on K if there is some* *continuous splitting and a continuous metric, and a uniform $\lambda < 1$ such that for all $(z, w) \in F \cap (K \times K)$, vectors in the unstable λ cone at z relate only to vectors in the unstable λ cone at w under $T_{(z,w)}F$. Further, it is backwards λ-contracting. In other words, if a vector v in the unstable cone relates to a vector v', then $|v| < \lambda|v'|$.*

Similarly, vectors in the the stable λ cone at w only come from the stable λ cone at z under $T_{(z,w)}F$, and are λ-contracting.

Lemma 4.11 (Cones and hyperbolic sets for relations) *Assume F is a relation, K is a compact hyperbolic set for F. Then F satisfies the cone condition on K.*

Proof Choose a nonzero vector (v_s, v_u) in the unstable cone at z. Suppose (v_s, v_u) relates to a vector (v'_s, v'_u) at w under the derivative relation. Then using the max norm,

$$|(v'_s, v_u)| < \lambda|(v_s, v'_u)|.$$

By hypothesis, $|v_s| < \lambda|v_u|$, so $|v'_u| > |v_s|$. Thus $|v'_s| < \lambda|v'_u|$, so the vector is in the unstable cone. Furthermore the derivative is expanding vectors, since $|(v_s, v_u)| = |v_u| < \lambda|v'_u| = \lambda|(v'_s, v'_u)|$. A similar proof holds for the stable cones. □

The obvious question here is whether the converse of the above lemma holds. In other words, if the cone condition holds on a compact set, then is the set hyperbolic? I believe that this converse is true.

4.4 Examples of hyperbolic sets

Example 4.12 A hyperbolic fixed point for a C^r relation is a hyperbolic set.

Example 4.13 For diffeomorphisms, the definition for relations is equivalent to the traditional definition. In other words, we have the following theorem;

Theorem 4.14 *If $f : R^n \to R^n$ is a diffeomorphism, and K is a compact invariant set under f, then K is a hyperbolic set for f by the traditional definition if and only if K is a hyperbolic set for the relation graph(f) under the relations definition.*

Proof By the Mather adapted norm, the diffeomorphism conditions imply that the new definition holds. The cone condition in the previous section implies that the converse holds as well. □

Example 4.15 (The case of homoclinic orbits) For diffeomorphisms, all transverse homoclinic orbits are embedded in hyperbolic sets. However, this is not always true for noninvertible maps and relations. Chapter 5 gives examples of noninvertible maps and relations with transverse homoclinic orbits, but for which the shadowing lemma does not hold. Since the next section shows that the shadowing lemma holds for all hyperbolic sets, these examples cannot ever be embedded in hyperbolic sets. Chapter 5 also gives sufficient conditions for homoclinic orbits to be hyperbolic sets. These conditions are satisfied, for example, by the homoclinic orbits in the delayed regulation model.

Another special case of homoclinic orbits which are always hyperbolic sets are *snap-back repellers* [15], defined as follows.

Definition 4.16 (Snap-back repellers) *For a noninvertible map f with repelling fixed point p, a snap-back repeller is a homoclinic orbit $\{z_k\}_{k\in I}$ for which every point in the orbit has $Df(z_k)$ an isomorphism.*

Namely, snap-back repellers are homoclinic orbits of repelling fixed points, which are contained in the zero-dimensional stable manifold.

Example 4.17 (Iterated function systems) As described in Chapter 2, an iterated function system is a relation which is the union of a finite number of smooth contractions $\{\omega_k\}_{k\in I}$. Under the assumption that for all x and $i \neq j$, $\omega_i(x) \neq \omega_j(x)$, an iterated function system forms a smooth relation. The whole space has a trivial splitting with no unstable directions. Thus the entire space has hyperbolic structure.

4.5 Shadowing

The shadowing lemma states that near a hyperbolic set, making small errors on each iteration still gives a reasonable picture of the dynamics. The shadowing lemma for diffeomorphisms is due to Bowen [5]. Here we give a proof of it for hyperbolic sets for smooth relations. The proof is functional analytic in nature. However, it seems to be conceptually simpler than the standard functional analytic proofs for diffeomorphisms, as the definition of hyperbolic sets in terms of a contraction allows the straight forward application of the contraction mapping theorem.

Notation: For $\gamma > 0$ and set S, the closed ball of radius γ is denoted by $B_\gamma(S)$.

Definition 4.18 (Pseudo-orbit) *A sequence $\{z_i\}_{i\in I}$ is called a δ-pseudo-orbit for relation F when $dist((z_i, z_{i+1}), F) < \delta$ whenever $i, i+1 \in I$. More elegantly in terms of relations, a δ-pseudo-orbit for F is an orbit of $B_\delta(F)$.*

Definition 4.19 (Shadow) *An orbit of F $\{w_i\}_{i\in I}$ ϵ-shadows a sequence $\{y_i\}_{i\in I}$ when for all $i \in I$, $dist(w_i, y_i) < \epsilon$.*

If in some set, every sufficiently small pseudo-orbit has a unique nearby shadow, the set is said to have the *shadowing property.* The following lemma says that hyperbolic sets have the shadowing property.

Theorem 4.20 (Shadowing) *If K is a hyperbolic set for F, then for any $\epsilon > 0$, there is a $\delta > 0$ such that any δ-pseudo-orbit in $B_\delta(K)$ is ϵ-shadowed by an orbit* *of F. If ϵ is small enough and the pseudo-orbit is bi-infinite, then its ϵ-shadow is unique. Further, if the pseudo-orbit is periodic, so is its shadow.*

The proof is conceptually simple. In summary, uniformly close to a pseudo-orbit near K, F is the graph of a contraction. Associated with F, there is a relation on the space of sequences. Near the pseudo-orbit, this sequence relation is the graph of a contraction from the space of sequences to itself. By the contraction mapping theorem, this contraction has a unique fixed point, which is the shadowing orbit we wanted to find. Here are the details of the argument.

Proof Assume we are given relation F and hyperbolic set K.

Given ϵ sufficiently small, there is a small enough δ so that if $z, w \in B_\epsilon(K)$ and $(z, w) \in B_\delta(F)$, then F is the graph of a Lipschitz contraction on $B_\epsilon(z, w)$.

Assume that $z^* = \{z_i\}_{i \in I}$ is a δ-pseudo-orbit in $B_\delta(K)$.

For sequence z^*, note that at each point there is an induced splitting, defined by $E^{s*}_{z^*} = \left\{E^s_{z_i}\right\}$ and $E^{u*}_{z^*} = \left\{E^u_{z_i}\right\}$. Thus there is an induced splitting for any sequence in $B_\epsilon(z^*)$.

For sequence $x^* \in E^{s*}_{z^*}$, define the norm $\|x^*\| = \sup_i |x_i|$.

Likewise, for $y^* \in E^{u*}_{z^*}$, let $\|y^*\| = \sup_i |y_i|$.

On $(x^*, y^*) \in E^{s*}_{z^*} \times E^{u*}_{z^*}$, use the norm $\|(x^*, y^*)\| = \max(\|x^*\|, \|y^*\|)$.

Let F^* be the relation induced by F. Namely, on the space of bi-infinite sequences in $B_\epsilon(z^*)$:

$w^* \overset{F^*}{\mapsto} \rho^*$ if and only if for all i, $w_i \overset{F}{\mapsto} \rho_{i+1}$

Lemma 4.21 *For any $\lambda' > \lambda$, F^* is the graph of a $Lip_{\lambda'}$ function f^* induced by f.*

Proof Assume $(x^*, y^*) \overset{F^*}{\mapsto} (\alpha^*, \beta^*)$. Then $(x_i, y_i) \overset{F}{\mapsto} (\alpha_{i+1}, \beta_{i+1})$, which implies $f(x_i, \beta_{i+1}) = (\alpha_i, y_{i+1})$. Thus there is a function f^* induced on sequences such that $f^*(x^*, \beta^*) = (\alpha^*, y^*)$.

To show that this function is $\mathrm{Lip}_{\lambda'}$, assume $(\xi^*, \eta^*) \overset{F^*}{\mapsto} (\theta^*, \mu^*)$. Thus $(\xi_i, \eta_i) \overset{F}{\mapsto} (\theta_{i+1}, \mu_{i+1})$.

By assumption, we know $\max(|\alpha_{i+1} - \theta_{i+1}|, |y_i - \eta_i|) < \lambda \max(|x_i - \xi_i|, |\beta_{i+1} - \mu_{i+1}|)$.

Taking the sup of both sides, we see that $\max(\|\alpha^* - \theta^*\|, \|y^* - \eta^*\|) \leq \lambda \max(\|x^* - \xi^*\|, \|\beta^* - \mu^*\|)$. Thus f^* is $\mathrm{Lip}_{\lambda'}$. □

Lemma 4.22 *F^* has a fixed point if and only if f^* has a fixed point.*

Proof Follows from the fact that $((x^*, y^*), (\alpha^*, \beta^*)) \in F^*$ if and only if $f^*((x^*, \beta^*)) = (\alpha^*, y^*)$. □

Lemma 4.23 *By the contraction mapping theorem, f^* has a unique fixed point.*

Proof The space of bi-infinite sequences on compact balls of radius ϵ along with the norm previously described is a Banach space. Thus we can use the contraction mapping theorem. □

From the above lemma, F^* has a unique fixed point. A fixed point of F^* is an orbit of F. Thus this fixed point sequence is the unique shadow of the pseudo-orbit z^*.

This completes the proof of the shadowing lemma. □

4.6 Stable manifold theorem for hyperbolic sets of relations

This section describes a generalization of the stable manifold theorem for hyperbolic sets which holds for noninvertible maps and relations.

The stable manifold theorem for hyperbolic sets of diffeomorphisms says that for every point in a hyperbolic set, the set of points with nearby forward orbits and the set of points with nearby backward orbits both form smooth manifolds. It was originally stated by Hirsch and Pugh [12]. In the case of relations, the nonuniqueness of orbits implies that this statement breaks down.

The delayed regulation model previously discussed is an example of a hyperbolic set for a noninvertible map such that there are multiple unstable manifolds to a point. At $a = 2.28$ the transverse homoclinic orbits form hyperbolic sets. However, within the hyperbolic set, the unstable manifold to the origin is not unique. By the same calculation which showed there was no invariant unstable direction at the origin, there are many curves which comprise the unstable manifold to the origin, as can be seen in Figure 4.1. Each of these curves follows one of the backwards orbits to the origin.

As the preceding example suggests, the stable manifold theorem for hyperbolic sets for relations is a theorem about orbits rather than about points. Since the shadowing lemma guarantees that these orbits are isolated, we lose any results about continuously varying manifolds. Although for relations stable and unstable manifolds correspond to orbits rather than to points, the new theorem is powerful in a different way. Namely, it shows that for each orbit in a hyperbolic set, there are stable and unstable manifolds, consisting of points with locally unique forward and backward orbits respectively. Since for a given point, orbits of a noninvertible map or relation are neither guaranteed to exist nor to be locally unique, this is a powerful result.

Definition 4.24 (Local manifolds for orbits) *Let $\{z_k\}_{k\in I}$ be an orbit for a smooth relation F on space Z. The stable and unstable manifolds to $\{z_k\}$ at z_i, denoted $W^s_{\{z_k\}}(z_i)$ and $W^u_{\{z_k\}}(z_i)$ are defined by:*

$W^s_{\{z_k\}}(z_i) = \{w \in Z :$ *there exists an infinite forward orbit* $\{w_k\}$ *through* w *such that* $w_k \to z_k$ *as* $k \to \infty\}$.

$W^u_{\{z_k\}}(z_i) = \{w \in Z :$ *there exists an infinite backward orbit* $\{w_k\}$ *through* w

such that $w_k \to z_k$ as $k \to -\infty\}$.

Theorem 4.25 (Stable manifolds for relations) *Let F be a smooth relation with compact hyperbolic set K. Assume $\{z_k\}_{k\in I}$ is an orbit of F contained in K.*

For sufficiently small ϵ, if $[i,\infty) \subset I$, the local stable manifold $W^s_{\{z_k\}}(z_i,\epsilon)$ is the graph of a Lipschitz function $E^s_{z_i} \to E^u_{z_i}$. Further, each point q_o in the local stable manifold has a locally unique forward orbit. In other words, there is only one forward orbit through q_o converging to the orbit $\{z_k\}$.

Similarly, for the unstable case, if $(-\infty, i] \subset I$, and ϵ is sufficiently small, then the local unstable manifold $W^u_{\{z_k\}}(z_i,\epsilon)$ is the graph of a Lipschitz function $E^u_{z_i} \to E^s_{z_i}$. Further, each point r_o in the local unstable manifold has a locally unique backward orbit.

For diffeomorphisms, the stable and unstable manifolds for points in hyperbolic sets are as smooth as the the diffeomorphism. Smoothness seems likely for stable and unstable manifolds for relations as well. Note that the estimates in the previous chapter for the fixed point case do not apply here, since there we assumed invariance of subspaces, and we do not assume that here.

Before beginning the proof, we develop some notation and reformulate the theorem in terms of this notation; assume F is a relation with hyperbolic set K, and $\{z_k\}_{k\in I}$ is an orbit in K.

For $(z,w) \in R^n \times R^n$, denote the projections to the two coordinates by π_1 and π_2; namely, $\pi_1(z,w) = z$ and $\pi_2(z,w) = w$.

Definition 4.26 (Forward shadow) *S^1_i is the set of pairs ϵ-shadowing forward for one iterate. i.e.*

$$S^1_i = \{(u,v) \in F : (u,v) \in B_\epsilon(z_i, z_{i+1})\}\,.$$

Equivalently, $S^1_i = F \cap B_\epsilon(z_i, z_{i+1})$.

S_i^k is the set of pairs which are connected by a k-forward shadow. Equivalently, in terms of composition of relations,

$$S_i^k = S_{i+k-1}^1 \circ \cdots \circ S_{i+1}^1 \circ S_i^1.$$

Likewise, for backward shadowing,

Definition 4.27 (Backward shadow) *S_i^{-1} is the set of pairs ϵ-shadowing backward for one iterate. Thus, $S_i^{-1} = F \cap B_\epsilon(z_{i-1}, z_i)$.*

S_i^{-k} is the set of pairs which are connected by a k-backward ϵ-shadow. In terms of composition of relations,

$$S_i^{-k} = S_i^{-1} \circ S_{i-1}^{-1} \circ \cdots \circ S_{i-k+1}^{-1}.$$

By definition $W^s(z_i) = \lim_{k\to\infty} \pi_1(S_i^k)$, and $W^u(z_i) = \lim_{k\to\infty} \pi_2(S_i^{-k})$. Thus the following lemma in terms of the new notation is equivalent to the stable manifold theorem for hyperbolic sets for relations.

Lemma 4.28 *Under the hypotheses of theorem 4.25, there is a constant $0 < \lambda < 1$ such that $\lim_{k\to\infty} \pi_1(S_i^k)$ exists and is the graph of a Lip_λ function $E_{z_i}^s \to E_{z_i}^u$. Each point in this limit set has a unique forward orbit converging to $\{z_k\}$. Also $\lim_{k\to\infty} \pi_2(S_i^{-k})$ exists and is the graph of a Lip_λ function $E_{z_i}^u \to E_{z_i}^s$. Each point in this limit set has a unique backward orbit converging to $\{z_k\}$.*

Proof

Here is the proof for the backward orbits case. Forward orbits case follows by symmetry.

Step 1 $\pi_2\left(S_i^{-k}\right) \subset \pi_2\left(S_i^{-(k-1)}\right)$.

This is because the set of points shadowing backward k iterates automatically must shadow backward for $k-1$ iterates.

Step 2 $\pi_2\left(S_i^{-k}\right)$ *is compact.*

This set is a projection of S_i^{-k}, which is compact since it is the composition of compact relations.

Step 3 *If δ is sufficiently small, then for every k and every $y \in E^u_{z_i}$, there exists x so $(x, y) \in \pi_2\left(S_i^{-k}\right)$.*

This follows from the following lemma on the composition of relations.

Lemma 4.29 *The composition of relations which are graphs of Lipschitz functions is a graph of a Lipschitz function. Specifically, if F is the graph of Lip_λ function f, G the graph of Lip_λ function g,*

$f : E_1^s \times E_2^u \to E_2^s \times E_1^u$, and

$g : E_2^s \times E_3^u \to E_3^s \times E_2^u$

then $G \circ F$ is the graph of a Lip_λ function $E_1^s \times E_3^u \to E_3^s \times E_1^u$

Proof of the lemma: By the Lipschitz Implicit Function Theorem. □

Step 4 $\lim_{k\to\infty} \pi_2\left(S_i^{-k}\right)$ *is the graph of a function from $E^u_{z_i}$ to $E^s_{z_i}$.*

From the previous step, $\pi_2\left(S_i^{-k}\right)$ is nonempty. Thus $\lim_{k\to\infty} \pi_2\left(S_i^{-k}\right) = \cap_{k>0}\pi_2\left(S_i^{-k}\right)$ is nonempty since it is the intersection of a nested sequence of compact sets. Further, using the same reasoning, for every $y \in E^u_{z_i}$, there is an $x \in E^s_{z_i}$ such that $(x, y) \in \lim_{k\to\infty} \pi_2\left(S_i^{-k}\right)$.

Before giving the final step in the proof, here is a useful estimate:

Lemma 4.30 *Assume $\{w_{-k}\}$ and $\{u_{-k}\}$ are backwards orbits such that for some sufficiently small $\gamma > 0$, and $i \le k < N$, dist(w_{-k}, z_{-k}) and dist$(u_{-k}, z_{-k}) < \gamma$. Then by continuity of the splitting we can rewrite $w_{-k} = (x_{-k}, y_{-k}), u_{-k} = (\xi_{-k}, \eta_{-k}) \in E^s_{z_{-k}} \times E^u_{z_{-k}}$. Then:*

$$\max(|x_i - \xi_i|, |y_{i-1} - \eta_{i-1}|) < \max(\lambda^k|x_{i-k} - \xi_{i-k}|, \lambda|y_i - \eta_i|).$$

Proof Since F is Lip_λ, $\max(|y_{i-1}-\eta_{i-1}|, |x_i-\xi_i|) < \max(\lambda|x_{i-1}-\xi_{i-1}|, \lambda|y_i-\eta_i|)$. Call this right hand quantity Δ.

Similarly, $|x_{i-1}-\xi_{i-1}| < \max(\lambda|x_{i-2}-\xi_{i-2}|, \lambda|y_{i-1}-\eta_{i-1}|)$.

Thus

$$|x_i-\xi_i| < \Delta < \max(\lambda^2|x_{i-2}-\xi_{i-2}|, \lambda^2\Delta, \lambda|y_i-\eta_i|).$$

But $\Delta > \lambda^2\Delta$, so

$$|x_i-\xi_i| < \Delta < \max(\lambda^2|x_{i-2}-\xi_{i-2}|, \lambda|y_i-\eta_i|).$$

Continuing inductively gives the desired inequality. □

Corollary 4.31 *If* (x_i, y_i) *and* (ξ_i, η_i) *both in* $\lim_{k\to\infty}\pi_2\left(S_i^{-k}\right)$*, then* $|x_i-\xi_i| < \lambda|y_i-\eta_i|$.

Proof Choose N so that $\lambda^N\epsilon < \lambda|y_i-\eta_i|$. By assumption, there is some (x_{i-N}, y_{i-N}) and (ξ_{i-N}, η_{i-N}) such that $(x_{i-N}, y_{i-N}, x_i, y_i)$ and $(\xi_{i-N}, \eta_{i-N}, \xi_i, \eta_i)$ $\in S_i^{-N}$.

From the preceding lemma,

$$\max(|x_i-\xi_i|, |y_{i-1}-\eta_{i-1}|) < \max(\lambda^N|x_{i-N}-\xi_{i-N}|, \lambda|y_i-\eta_i|) < \lambda|y_i-\eta_i|.$$

□

Step 5 $\lim_{k\to\infty}\pi_2\left(S_i^{-k}\right)$ *is the graph of a* Lip_λ *function. Every point of the limit set has a unique backward orbit converging to the original orbit.*

That the limit set is the graph of a function follows from the previous step. That the function is Lipschitz follows from the above corollary.

To show that points have unique backwards orbits in the limit set, consider the behavior of the inverse relation when restricted to the limit set. From the last line of the proof of the above corollary, the inverse relation is a contraction. A relation which is a contraction is a function. Thus the inverse relation when restricted to the limit set is actually a function. From this it is clear that points in the limit set have unique backward orbits and that the orbits converge to $\{z_k\}$.

This final step completes the proof of the stable manifold theorem for hyperbolic sets for relations. □

4.7 Iterated function systems example

As an illustration of the ideas of this chapter, consider once again iterated function systems. As mentioned in Section 2.5, if $\{\omega_k\}_{k\in I}$ are a finite number of smooth contractions, and for all x and $i \neq j$, $\omega_i(x) \neq \omega_j(x)$, then this iterated function system forms a smooth relation $F = \cup\omega_k$, and every compact set is hyperbolic.

The stable manifold theorem says that every point z_o with an infinite forward orbit $\{z_k\}_{k\geq 0}$ has a neighborhood in which every point w has a forward orbit converging to the orbit $\{z_k\}_{k\geq 0}$. In fact, on a compact space, this neighborhood turns out to be the whole space, the orbit depending only on the sequence of contractions chosen. The theorem also says that the unstable manifold for a backward orbit is zero-dimensional. The shadowing lemma says that every sufficiently small bi-infinite pseudo-orbit has a unique shadow.

Using the above, we can recover information about which sequence of contractions in F converges to a given point, as in Barnsley's Chaos Game [4]. Let $(x, y) \in F$. Then $y = \omega_j(x)$ for some j. Notice that for sufficiently small δ, all points in $F \cap B_\delta(x, y)$ are also of the form $(z, \omega_j(z))$ for the same j. Thus for a sufficiently small pseudo-orbit, we can recover the bi-infinite sequence of contractions from F which give the shadowing orbit. Also, to every backward sequence of contractions, there corresponds a unique point with the sequence corresponding to its backward orbit. However, there may be many forward and backward orbits through such a point.

In [4], there is a proof of a version of the shadowing lemma for iterated function systems which are are the union of invertible contractions. Here, we do not assume that contractions are invertible, but the main difference is in the framework and approach to looking at iterated function systems.

Chapter 5

Homoclinic Orbits

One of the basic examples of a hyperbolic set for diffeomorphisms is the closure of a transverse homoclinic orbit. Such an orbit implies the existence of the famous *homoclinic tangle*, in which as early as Poincaré, mathematicians observed complicated behavior. Using the shadowing lemma it is possible to show the precise nature of the complicated behavior nearby such an orbit.

For noninvertible maps and relations, transverse homoclinic orbits are not necessarily embedded in hyperbolic sets. This chapter gives the basic definitions and implications of transverse homoclinic orbits for diffeomorphisms. It then gives examples of the manner in which these implications fail for noninvertible maps and relations. It ends with a sharp condition for when the closure of transverse homoclinic orbits of relations are hyperbolic sets.

5.1 Definitions and background

The following definitions follow the standard literature, such as [23] or [28], as closely as possible. Extra care is taken so that the definitions still make sense for noninvertible maps.

Definition 5.1 (Homoclinic point) *If a smooth relation has a hyperbolic fixed point with global stable and unstable manifolds W^s and W^u, then a homoclinic point is a point in $W^s \cap W^u$.*

Definition 5.2 (Homoclinic orbit) *A bi-infinite orbit is called a homoclinic orbit if it converges to the fixed point both forwards and backwards. i.e. for a homoclinic orbit $\{z_k\}$, $\lim_{k\to\infty} z_k = \lim_{k\to-\infty} z_k = p$.*

Through an arbitrary point in a noninvertible map or relation, there may be multiple orbits or no orbits. However, the following lemma says that through each homoclinic point, there exists a homoclinic orbit.

Lemma 5.3 *Assume there is a smooth relation with hyperbolic fixed point p and a homoclinic point q. Then q is contained in a homoclinic orbit. Conversely, every point in a homoclinic orbit is a homoclinic point.*

Proof This follows from the definitions of the stable and unstable manifolds and homoclinic point. □

Definition 5.4 (Transverse homoclinic point) *Assume a smooth relation has a hyperbolic fixed point with stable and unstable manifolds W^s and W^u. A transverse homoclinic point is a point at which W^s and W^u intersect transversally.*

Transverse intersection is a property of smooth manifolds. Thus the above definition makes sense only when the stable and unstable manifolds are locally smooth. When defined, it gives a conceptually simple geometric condition occurring near one point. Since transverse intersection of manifolds is stable under perturbation, it is possible to check the definition computationally, assuming it is possible to compute the stable and unstable manifolds.

5.2 Homoclinic tangle

Global stable and unstable manifolds for diffeomorphisms are immersed submanifolds of the ambient space. Thus transversality of homoclinic orbits is always well defined. Furthermore, the existence of a transverse homoclinic point implies recurrent behavior, as described in the following theorem.

Theorem 5.5 (Homoclinic tangle for diffeomorphisms) *For diffeomorphism f, let q be a transverse homoclinic point for a hyperbolic fixed point p, and let U be a neighborhood containing p and q. Then U contains a compact hyperbolic invariant set K, and there is some n such that on K, f^n is topologically conjugate to the shift map on two symbols.*

Also, in any neighborhood of the closure of the orbit of q, there is a hyperbolic invariant set for f and an n such that f is topologically conjugate to a subshift of finite type on n symbols.

The key observation of the proof is that there exists a hyperbolic structure on the closure of the orbit of q. (The closure of the orbit of q is equal to the orbit union p.) Precisely, setting $E^s_x = T_x W^s$ and $E^u_x = T_x W^u$, it is possible to show that the homoclinic orbit has the proper expanding and contracting behavior. Once this is verified, the proofs of the two statements in the above theorem follow easily using the shadowing lemma.

There are many things that can go wrong with the theory of transverse homoclinic points for noninvertible maps. In contrast to the diffeomorphism case, for noninvertible maps, the global stable and unstable manifolds for noninvertible maps are not necessarily smooth, as described in Chapter 3. Thus transversality of homoclinic points is not always well defined.

5.3 Counterexamples

Even when tranversality of homoclinic points is well defined for noninvertible maps and relations, the homoclinic tangle theorem above does not hold for noninvertible maps. The first example below shows the existence of a noninvertible map with a transverse homoclinic point with no recurrent behavior. Thus the shadowing property fails to hold. The other examples show the existence of transverse homoclinic orbits for relations with infinitely many points shadowing arbitrarily small pseudo-orbits. Thus the uniqueness of orbits for the shadowing property fails.

Example 5.6 (Homoclinic orbit with no shadowing orbits) We start with an orientation-preserving diffeomorphism $f : R^2 \to R^2$ with a transverse homoclinic orbit. We then introduce some standard terminology for diffeomorphisms, to make precise the statement that near a point q in the homoclinic orbit, all points on one side of the stable manifold never return near q under f. Namely, these points end up following the branch of the unstable manifold which never returns to q. Using this fact, we perturb the diffeomorphism in a neighborhood of q in such a way that all the points in the neighborhood map to the same side of the stable manifold. Under this new noninvertible map, no points in the neighborhood of q return to near q.

5.3.1 Pips, lobes, and transport

The terminology introduced here for diffeomorphisms of R^2 follows [28]. First, the idea that r is a primary intersection point of W^s and W^u.

Definition 5.7 (Primary intersection point, or pip) *Assume f is a diffeomorphism with hyperbolic fixed point p. Homoclinic point r is a primary intersection point, or pip, when the segments of W^s and W^u joining p to r intersect only at p and r.*

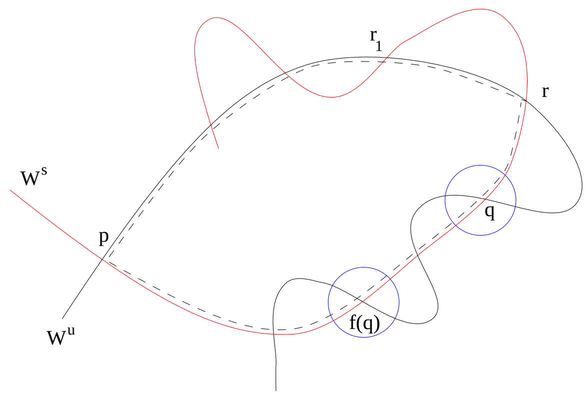

Figure 5.1: Transverse homoclinic orbit of an orientation preserving diffeomorphism. The dotted line marks the pseudoseparatrix. The regions near q and $f(q)$ mark the perturbed domain and range respectively.

Definition 5.8 (Lobe) *Let r_1 and r_2 be two adjacent pips. Precisely, there are no pips between them along the segment joining p to r_1 either along W^s or along W^u. The lobe is a region bounded by the segments joining r_1 to r_2 along W^s and along W^u*

It is possible to bound a region using the segments of W^s and W^u joining p to r. This region is called a *pseudoseparatrix*. Assuming that there is only pip r_1 between r and $f^{-1}(r)$, it is possible to use the r, r_1 and $r_1, f^{-1}(r)$ lobes to classify the movement of points in and out of the pseudoseparatrix.

Lemma 5.9 (Turnstile lobes) *If pip r_1 is the only one between r and $f^{-1}(r)$, then r, r_1 lobe contains all point entering the pseudoseparatrix in one iterate, and the $r_1, f^{-1}(r)$ lobe contains all the points leaving the pseudoseparatrix in one iterate.*

5.3.2 Perturbing the diffeomorphism

Using the notation developed above, assume that r is a pip, and that there is one pip r_1 between r and $f^{-1}(r)$. Thus the r, r_1 lobe contains all points entering the

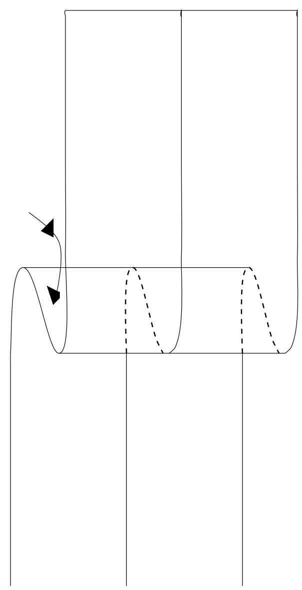

Figure 5.2: A cubic map folds the plane over itself.

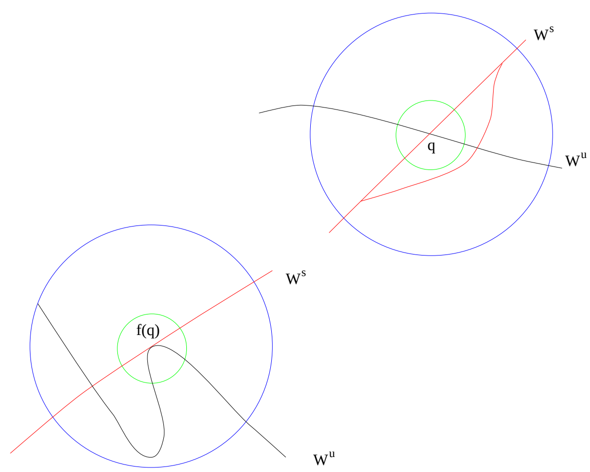

Figure 5.3: A look at the change of stable and unstable manifolds near q and $f(q)$. Note that there is an extra curve added to W^s.

pseudoseparatrix made with r in one iterate. Let $q = f(r_1)$. By the above lemma, and the fact that points cannot map across the W^u, in a neighborhood of q, points outside the pseudoseparatrix never return near q under forward images of f. See Figure 5.1

Choose a neighborhood of q. Smoothly perturb f in this neighborhood to get a new map g. Do this in such a way that outside the neighborhood, the map is equal to f, and inside a slightly smaller neighborhood, all points not on $W^s(f)$ map to the outside of the pseudoseparatrix, and all points on $W^s(f)$ map onto $W^s(f)$. This can be done, for example, with a cubic map. See Figure 5.2.

In a neighborhood of q, the unstable manifold for g is the same as the unstable manifold for f. By mapping points on $W^s(f)$ to other points on $W^s(f)$, we assure that the stable manifold is locally near q the same as before the perturbation. Away from q, but in the perturbed neighborhood, there is actually an extra portion of $W^s(g)$ formed as the preimage under the cubic map, but this does not affect the behavior sufficiently near q. See Figure 5.3. $W^s(g)$ and $W^u(g)$ still intersect transversally at q. Since all points near q map outside the pseudoseparatrix under g, no point near q ever returns near q. Thus there is no chaotic behavior near the homoclinic orbit containing q. Note that for all δ, there are δ-pseudo-orbits containing q. Thus shadowing does not hold.

5.3.3 Analysis of the example

One difference between a diffeomorphism and the map in this construction is that not every point in the orbit is a transverse homoclinic point. A second feature of this example is that the derivative map is singular on the tangent to the unstable manifold. The proof that a transverse homoclinic orbit for a diffeomorphism has a hyperbolic structure relies on the fact that the tangent to the unstable manifold is eventually expanding. Clearly that condition fails to hold here.

When these two differences described above are absent, the homoclinic orbit of a noninvertible map can be embedded in a hyperbolic set. This is made precise in the next section.

Example 5.10 (Homoclinic orbit with nonunique orbits) Here is a relation on R^1 with a transverse homoclinic orbit, points of which also form a periodic orbit. Again in this case, the shadowing property fails, implying that the orbit was not contained in a hyperbolic set. Let F be a relation on R^1 with the following properties:

1. $(0,0) \in F$; *i.e.* 0 is a fixed point.

2. Locally near 0, the relation is equal to the line $(x, a^2x), a < 1$. This means that 0 is a contracting fixed point.

3. $(0,s) \in F$ and $(s,0) \in F$. This means that 0 and s are points in a period two orbit. The points are also contained in a homoclinic orbit to the fixed point.

4. Locally near both $(0,s)$ and $(s,0)$, the relation is equal to a line $(x, \frac{1}{a}x)+$ the point. Thus the period two orbit through the fixed point is expanding. Since the unstable manifold is zero-dimensional, the homoclinic orbit containing these two points is transverse.

See Figure 5.4. The composition of F near $(0,s)$ with F near $(s,0)$ gives a portion of F^2 near $(0,0)$, which is equal to $(x, \frac{1}{a^2}x)$. Composing this with F near $(0,0)$ gives a portion of F^3 near $(0,0)$ which is equal to the line (x,x), the identity map. From this we see that every point in a neighborhood of the fixed point has bi-infinite orbits shadowing the orbit $...0, s, 0, 0, s, 0, ...$ Thus uniqueness required for the shadowing property fails.

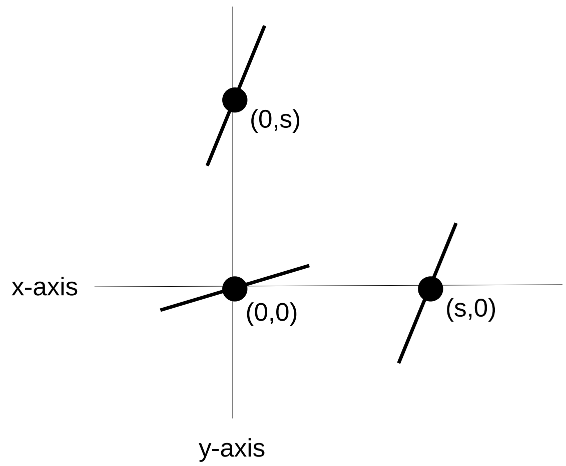

Figure 5.4: The relation described in Example 5.10 near the homoclinic orbit.

In this example, points of the homoclinic orbit are also contained in a periodic orbit. Further, the fixed point is contracting as a fixed point, but as a point in a periodic orbit, it is expanding. Since the homoclinic orbit relates both forward and backward to the fixed point, there is no way to make up eventually for the expansion around the orbit.

Example 5.11 (Homoclinic orbit with incompatible directions) Here is another example of a relation on R^1 with a homoclinic orbit which does not satisfy the uniqueness part of the shadowing property. Unlike the previous example, the the derivative of the periodic orbit and the derivative of the fixed point are both contracting. However, the tangent directions are not maps in compatible coordinate systems. Let F be a relation on R^1 with the following properties:

1. $(0, 0) \in F$; *i.e.* 0 is a fixed point.

2. Locally near 0, the relation is equal to the line (x, ax), $0 < a < 1$. This means that 0 is a contracting fixed point.

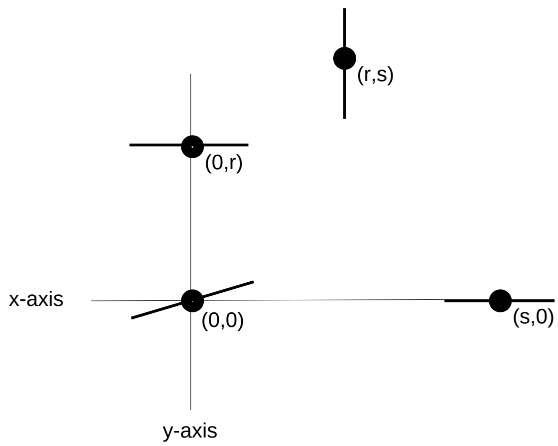

Figure 5.5: The relation described in Example 5.11 near the homoclinic orbit.

3. $(0, r), (r, s), (s, 0) \in F$. Thus $\{0, r, s\}$ are contained in a period three orbit. They are also contained in a homoclinic orbit to the fixed point.

4. Locally near $(0, r)$ and $(s, 0)$, the relation is equal to a line $(x, 0)+$ the point. Locally near (r, s), the relation is equal to a line $(0, y) + (r, s)$.

See Figure 5.5. Points near 0 relate to points near r. r relates to points near s. Points near s relate to 0. Thus, under the third iterate through the period three orbit, points near 0 relate to 0, which is also a contracting relation near 0. Notice that near r, there are no other points in the unstable manifold of 0, so the homoclinic orbit is a transverse homoclinic orbit. However, not only is $\{0, r, s\}$ part of a period three orbit, but also for any small x, $\{0, r, s + x\}$ is contained in a periodic orbit as well. Thus there is no uniqueness for the shadowing of pseudo-orbits.

This example cannot be explained by the fact that the points of the homoclinic orbit formed a periodic orbit. This choice was only for simplicity. It is possible to make an example with a similar behavior, but for which the homoclinic orbit converges to the fixed point in both directions, rather than consisting of points in a periodic orbit. The problem here is the incompatibility of coordinates. No matter how we change the metric, the relation from r to s is not the graph of a contraction. This shows that it is necessary to require compatible coordinates for the relation, rather than looking at a high iterate of the relation in the neighborhood of the fixed point.

5.4 Conditions for hyperbolicity

This section describes conditions for a transverse homoclinic orbit to be embedded in a hyperbolic set. The first version is geometric. The second version is stronger and well-defined for all homoclinic orbits of noninvertible maps. It is the condition from the work of Steinlein and Walter [27]. Their work lacks the examples such as in the previous section to make the results sharp. The third version is an extension of the section version to a condition for relations. All results have the disadvantage that checking the specified conditions is in general computationally infeasible.

Hypotheses: For the first two theorems, let p be a hyperbolic fixed point for a map, and let $\{z_k\}$ be a homoclinic orbit to p.

Theorem 5.12 (Homoclinic orbits for maps) *If every point in the orbit is a transverse homoclinic point, and at each point the derivative is an isomorphism on the tangent to the unstable manifold, then the closure of the orbit is hyperbolic.*

Theorem 5.13 (Homoclinic orbits for maps, stronger version) *Let $\{z_k\}$ be a homoclinic orbit. Then there exists a sufficiently large N such that for all $m < -N$ and $n > N$, $z_m \in W^u_{loc}$, and $z_n \in W^s_{loc}$. If $Df^{n-m}(z_m)$ is injective on $T_{z_m}W^u$ and maps $T_{z_m}W^u$ to a subspace transversal to $T_{z_n}W^s$, then the closure of $\{z_k\}$ is hyperbolic.*

Note that this theorem is more general than the previous one.

Definition 5.14 (Inverse relation) *For the relation F, the inverse relation, denoted F^* is equal to:*

$$\{(z, w) : (w, z) \in F\}.$$

Theorem 5.15 (Homoclinic orbits for relations) *Let p be a hyperbolic fixed point for a smooth relation F on R^N, and let $\{z_k\}$ be a homoclinic orbit to p.*

Denote the tangent relation along the orbit by $T(j, k)$. Precisely, $T(j, k) = TF(z_{k-1}, z_k) \circ \ldots \circ TF(z_{j+1}, z_{j+2}) \circ TF(z_j, z_{j+1})$.

Assume that there exists some local neighborhood of p small enough that for all $z_m \in W^u_{loc}$ and $z_n \in W^s_{loc}$, the following conditions hold:

1. **Transversality and compatibility:** *In the local neighborhood of the fixed point, define $E^s_{z_n} = T_{z_n}W^s$ and $E^u_{z_m} = T_{z_m}W^u$. For each k, the set of vectors relating to $E^s_{z_n}$ under the linear relation $T^*(n, k)$ is a subspace of the same dimension as $E^s_{z_n}$. If $z_k \neq p$, call this subspace $E^s_{z_k}$. Likewise, the set of vectors relating to $E^u_{z_m}$ under $T(m, k)$ is a subspace of the same dimension as $E^u_{z_m}$. If $z_k \neq p$, call this subspace $E^u_{z_k}$. For each pair of points in the homoclinic orbit, $T(j, k)$ is a graph of a linear function in the skew coordinates: $E^s_{z_j} \times E^u_{z_k} \to E^s_{z_k} \times E^u_{z_j}$.*

2. **Nondegeneracy:** *Within the set of points in $\overline{\{z_n\}}$, none comprise periodic orbits, aside from p being a fixed point.*

Then the closure of the homoclinic orbit is hyperbolic.

This theorem implies the previous one; assuming the hypotheses of the previous theorem, if $T_{z_m}W^u$ maps injectively, then its image is always a subspace of the same dimension. Since these images are eventually transverse to $T_{z_n}W^s$, it must always be transverse to inverse images of $T_{z_n}W^s$, and the inverse image $T_{z_n}W^s$ under the derivative must also be of the same dimension as $T_{z_n}W^s$. Injectivity of the unstable subspaces also implies that the derivative is the graph of a linear contraction in the skew coordinate system. Furthermore, the nondegeneracy condition always holds for maps.

Here are two lemmas we will use in the proof of theorem 5.15.

Lemma 5.16 (Continuity of subspaces) *As* $z_k \to p$, $E^s_{z_k} \to E^s_p$ *and* $E^u_{z_k} \to E^u_p$.

Proof By the proof of the stable manifold theorem, in some sufficiently small neighborhood, $F^k \to F^\omega$ in C^k norm. Since $E^u_{z_{k+1}}$ is the image under the tangent relation of $E^u_{z_k}$, $E^u_{z_n} \to E^u_p$. □

Note that rather than the above lemma, the proof of this theorem for diffeomorphisms uses the λ-lemma. Since $F^k \to F^\omega$ in C^k norm, and since F^ω projects to W^u, I believe that the λ-lemma is true for relations as well.

Lemma 5.17 *There exists* $\lambda < 1$ *such that for* $v_s \in E^s_{z_k}$ *and* $v_u \in E^s_{z_k}$ $|T(k, k+j)v_s| < C\lambda^j |v_s|$ *and* $|T^*(k, k-j)v_u| < C\lambda^k |v_u|$.

Proof The two inequalities are well defined, since by the hypotheses, TF is a function on stable subspaces and TF^* is a function on unstable subspaces. At p, the derivative is contracting on the stable subspace; backwards, it is contracting on the unstable subspace. Thus near enough to p, there is a $\lambda < 1$ such that the inequalities are true with $C = 1$, since TF is continuously varying, and the subspaces are continuously varying as well. There are only finitely many points outside any neighborhood of the p. Thus C is just an upper bound for the value away from p. □

Proof of theorem 5.15

Because we are dealing with relations, it is possible for a point in the homoclinic orbit other than p to be related to p. It is also possible for p to be related to a another point in the homoclinic orbit. However, by the nondegeneracy hypothesis, in one direction, z_n converges to p only in the limit. Assume that this happens for $n \to -\infty$. The proof of the other case follows by symmetry.

Now we change the metric on the stable and unstable subspaces. Let L be such that $C\lambda^L < 1$. Pick $1 > \lambda' > \lambda$.

At the point $z_n \neq p$, define the new metric:

$$|v_s|' = \Sigma_{k=0}^{L} \lambda'^{-k} |T(n, n+k) v_s|$$

$$|v_u|' = \Sigma_{k=0}^{L} \lambda'^{-k} |T^*(n-k, k) v_u|.$$

This is well-defined since the derivative forward is invariant on the stable subspaces and is a map when restricted to the stable subspaces. Also, the derivative backward is invariant on the unstable subspaces, and is a map when restricted to unstable subspaces.

At p,

$$|v_s|' = \Sigma_{k=0}^{L} \lambda'^{-k} |TF_{(p,p)}^{k} v_s|$$

$$|v_u|' = \Sigma_{k=0}^{L} \lambda'^{-k} |TF_{(p,p)}^{*k} v_u|.$$

The coordinate system is continuous, since the derivative and the directions are continuously varying. Now other than at p, if (v_s, v_u) relates to (v_s', v_u'), then $|(v_s', v_u)|' = \max(|v_s'|', |v_u|') < \lambda' |v_s, v_u'|'$.

Let J be such that z_J relates to p. Now we alter the metric so that the derivative relation from z_J to p is a contraction in skew coordinates.

We know that $TF(z_J, p)$ is still the graph of a linear function $E_{z_J}^s \times E_p^u \to E_p^s \times E_{z_J}^u$, $(v_s', v_u) = (av_s + bv_u', cv_s + dv_u')$. We know $c = 0$ since $E_{z_J}^s$ is the inverse image of E_p^s. Also, $|a|' < 1$, by the choice of metric. Choose $1 > \mu > \lambda'$, and let $M = \mu \max(|b|', |d|')$. Now define a new metric on $E_{z_J}^u$ by $|v_u|'' = M|v_u|'$. In the new metric, the derivative relation is the graph of a μ-contraction at (z_J, p), but not at (z_{J-1}, z_J). So define a new metric on $E^u_{z_{J-1}}$ by $|v_u|'' = \frac{\mu M}{\lambda'}|v_u|'$. Proceed inductively, and let the new metric at $E^u_{z_{J-k}}$ be $|v_u|'' = \frac{\mu^k M}{\lambda'^k}$. We only need to do this a finite number of times, since for a finite k, $\frac{\mu^k M}{\lambda'^k} < 1$. Thus we did not effect the continuity of the metric. This completes the proof. □

For snap-back repellers, the assumption that the derivative is an isomorphism makes these conditions automatically true. Correspondingly, there are also *snap-forward attractors* for relations which are hyperbolic sets, being orbits in the zero-dimensional unstable manifold of an attracting hyperbolic fixed point of a relation with derivative relation an isomorphism.

In general, the conditions in the above theorem would be difficult if not impossible to check. However, for the delayed regulation map, the map is invertible except on the x-axis. Since the unstable manifold is transverse to the stable manifold at each homoclinic point on the x-axis, the same must be true for all points in the homoclinic orbits. We only need to verify that the map is an isomorphism on the tangent to the unstable manifold at the homoclinic points on the x-axis. This is clearly true since $Df(q_1, 0)(\xi, 1) = (1, a(1 - q_1))$.

Chapter 6

Conclusion

6.1 Summary of results

In order to understand the dynamics of noninvertible smooth maps, we have considered in this thesis, the more symmetric iterated smooth relations. Namely, a smooth relation on R^n is an n-dimensional submanifold of R^{2n}. By iteration, we mean that if z and w in R^n, and the pair (z, w) is within the relation, then w is an iterate of z. Since the inverse of a smooth relation is also a smooth relation, this is a symmetric approach to forward and backward iteration.

Since iteration of relations is well defined, the concept of fixed point makes sense as well. If the tangent plane at a fixed point is the graph of a contraction which is invariant on stable and unstable subspaces, we call this fixed point hyperbolic. Note that this contraction has its domain and range in some *skew* coordinates, which are not relevant to the dynamics of the relation. In Chapter 3, we proved the local stable manifold theorem for a hyperbolic fixed point of a smooth relation.

The idea of looking at the relation as the graph of a contraction in some skew coordinates applies not only to fixed points but also to compact sets. In Chapter 4, we used this idea to develop the theory of hyperbolic sets for smooth relations. If there are continuous coordinates and a continuous metric on a compact set such that at every point in the set, the smooth relation is the graph of a contraction in terms of these coordinates, then we say that the set is hyperbolic. For diffeomorphisms, this is equivalent to the standard definition. This is due to the equivalence between the existence of invariant stable and unstable subspaces and the existence of stable and unstable cones. For general smooth relations, the new definition allows for the possibility of stable and unstable subspaces which do not map invariantly under the derivative.

From this definition of hyperbolic sets, we proved the shadowing lemma and a version of the stable manifold theorem for hyperbolic sets. In the diffeomorphism case, the stable manifold theorem for hyperbolic sets is a theorem about stable and unstable manifolds to points. However, since for noninvertible maps and relations, there are many orbits through each point for a relation, the stable manifold theorem is instead a theorem about forward and backward orbits.

One of the principle examples of a hyperbolic set for a diffeomorphism is the closure of a transverse homoclinic orbit to a hyperbolic fixed point. In Chapter 5, we gave several examples to show that the closure of a transverse homoclinic orbit is not always a hyperbolic set. We then generalized Steinlein and Walther's conditions for noninvertible maps; we formulated conditions for when the closure of a transverse homoclinic orbit of a hyperbolic fixed point for a smooth relation is a hyperbolic set.

6.2 Some open questions

To conclude, the following section lists some open questions related to the work in this thesis. This list is by no means comprehensive.

6.2.1 Global manifolds

As described in Chapter 3, nothing is known about global stable and unstable manifolds for maps and relations. In general, they seem to be as complicated as arbitrary sets. Can anything in fact be said about the underlying structure of these sets. Otherwise, what sorts of conditions guarantee some structure?

6.2.2 Adapted norms

If a set is hyperbolic for an iterate of a diffeomorphism, then it is automatically hyperbolic for the diffeomorphism. This comes from the fact that eventually expanding and contracting subspaces are immediately expanding for some adapted metric. However, for noninvertible maps and relations, it is not clear that this is true. The definition of hyperbolic sets used in this thesis always assumes immediate expansions and contractions. Is it possible to infer such a metric from the behavior of higher iterates? The adapted metric for diffeomorphisms comes from a weighted sum separately on the expanding and contracting subspaces; it is not clear how to renorm when there is no invariance of subspaces. In addition, looking at iterates of a noninvertible map or relation may be the wrong approach, since in general, F a smooth relation does not imply that F^k is a smooth relation. However, we make the following conjecture:

Conjecture 6.1 (Adapted norm for relations) *Assume F is a smooth relation, and K is a compact set such that every point in K has a bi-infinite orbit in*

K. Then if for some $m > 1$, F^m is a smooth relation, and if K is hyperbolic for F^m, then K is hyperbolic for F.

6.2.3 Homoclinic orbits

The previous chapter gave several examples for which transverse homoclinic orbits are not embedded in hyperbolic sets. It also gave some conditions for homoclinic orbits to have hyperbolic structure. Generically, does the existence of a transverse homoclinic point imply the existence of a hyperbolic structure on the homoclinic orbit? Transversality of intersection is not always well defined. Is it well defined generically? For noninvertible maps, in order for the unstable manifold to lose its injectivity and its local smooth structure, it must intersect the set of points for which derivative is singular. However, this is a codimension one set. Thus for high dimensions, it is plausible that this will happen generically. From this, here is a conjecture.

Conjecture 6.2 *For noninvertible maps and relations in high enough dimensions, transverse homoclinic points are not generically embedded in hyperbolic sets.*

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