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HYPERBOLIC SETS FOR NONINVERTIBLE MAPS AND RELATIONS

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Abstract. This paper develops the theory of hyperbolic sets for relations, a generalization of both noninvertible and multivalued maps. We give proofs of shadowing and the stable manifold theorem in this context.

1. **Introduction.** Not all iterated maps arising in natural systems have the property that time is reversible. In other words, there may be multiple points which map to the same point under iteration. This is evident in one-dimensional dynamics; for example, the frequently studied logistic map $x_{n+1} = \mu x_n(1 - x_n)$ is noninvertible. Such maps also arise in higher dimensional models. Examples of noninvertible maps have been studied in the contexts of population dynamics [3], time one maps of delay equations [13], control theory algorithms [1, 10, 9], neural networks [21], and iterated difference methods [19]. There have been relatively few attempts to classify the general theory of iterated noninvertible maps. Of particular note are the works of Hale and Lin [13] and Marotto [20] on homoclinic orbits, and the work of Steinlein and Walther [34, 33], and the textbook of Lani-Wayda [18] for hyperbolic sets in Banach spaces.

Aside from noninvertible maps, another useful generalization of the theory of diffeomorphisms is to the study of iterated multivalued dynamical systems, in which one point may map to multiple points forwards in time. Multi-valued maps have been used for a rigorous mathematical analysis of numerical methods for Conley index theory to give a computer-aided proof of the existence of chaos in the Lorenz equations [24], as well as in other related contexts [25].

In the current work, we develop a theory of hyperbolic sets for which shadowing and the stable manifold theorem hold. Rather than exclusively studying one or the other of the above concepts, we look at a generalization of both iterated noninvertible and multivalued maps, namely relations (Definition 4.4). If f is a diffeomorphism, then so is f^{-1}. Thus forward and backward iteration are symmetric concepts, a symmetry which is used extensively in the study of diffeomorphisms. In contrast, the inverse of a noninvertible map is a multivalued map, and the inverse of a multivalued map is a noninvertible map; in each case, there is a qualitative distinction between forward and backward iteration. However, the inverse of a relation is another relation, so the symmetry of iteration is restored. Our definitions and proofs of hyperbolic sets capitalize on this symmetry.

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The theory of hyperbolic sets for relations is new, but the theory of iterated relations can be seen in a number of previous works. McGehee [23] and Akin [2] laid down the framework for studying iterated relations. McGehee and the current author [22] gave a proof of the stable manifold in this context. Langevin et al developed the concept of entropy for relations [11, 16, 17]. In [30], Sintoff showed the parallels between the theory of iterated relations and the logic of iterative programs. Bullet and Penrose [6, 7] used relations, which they call correspondences, to study holomorphic dynamics.

In the context of relations, it is not possible to define a hyperbolic set exactly as in the diffeomorphism case. Since two points may map to the same point, it is too strict to assume that stable and unstable subspaces are invariant under the derivative map. Section 3 contains an example illustrating this problem. To avoid the assumption of invariance, the definition of hyperbolic sets for relations is a condition equivalent to the geometric condition of existence of stable and unstable cones. Section 4 illustrates the ideas of hyperbolicity for relations in the simple case of a hyperbolic fixed point, and Section 5 contains the formal definitions. Section 6 contains proof of equivalence between hyperbolicity and the cone condition. Section 7 contains a proof of the shadowing lemma. For relations, it is no longer possible to talk about a stable and unstable manifold to a point, since there may be many orbits to the same point, and thus many manifolds. However, there do exist stable and unstable manifolds for specified forward and backward orbits. Robustness and continuous change of manifolds then follow immediately. These results are described in Section 8. Finally, Section 9 contains some examples of hyperbolic sets.

2. **Background.** For a diffeomorphism, at each point in a hyperbolic invariant set, there is a continuous splitting of the tangent space into stable and unstable subspaces. These subspaces map invariantly under the derivative; on the stable subspaces the derivative is eventually uniformly contracting, and on the unstable subspaces the derivative is eventually uniformly expanding. In other words, we have the following definition.

Definition 2.1 (Hyperbolic sets for diffeomorphisms)**.** *A compact invariant set K for a diffeomorphism f is a hyperbolic set if the following conditions hold:*

1. *There is a continuous splitting of the tangent space into stable and unstable subspaces,*
$$T_xR^n = E^s_x \times E^u_x, \ \ x \in K.$$
2. *The derivative is invariant for the splitting,*
$$Df(x)E^s_x = E^s_{f(x)},$$
$$Df(x)E^u_x = E^u_{f(x)}.$$
3. *The derivative is eventually contracting on the stable subspaces and eventually expanding on the unstable subspaces. That is, for $C > 0$, $\lambda < 1$,*
$$|Df^n(x)v| < C\lambda^n|v|, \ \ v \in E^s_x,$$
$$|Df^n(x)v| > C^{-1}\lambda^{-n}|v|, \ \ v \in E^u_x.$$

By a theorem due to Mather [31], there is a continuous *adapted* metric such that $C = 1$, and thus the derivative contracts stable subspaces and expands unstable subspaces under just one iterate.

Hyperbolic sets for diffeomorphisms have been carefully studied, and there are many theorems describing the dynamics on such sets. Thus we would like to generalize the definition and corresponding theorems to relations. In the following sections, we justify modifications of the original definition by looking at what happens in the case of noninvertible maps. The same statements would apply to multivalued maps with time reversed.

3. **Why have a new definition?** For noninvertible maps, we cannot just apply the definition of hyperbolic sets as it stands. The requirement that stable and unstable subspaces map invariantly is too stringent. Trivially, if the derivative is singular, then we can only expect $Df(x)E^s_x \subset E^s_{f(x)}$. More subtly, we cannot expect the unstable subspaces to map invariantly either, since there may be multiple points mapping to the same point, as shown in the following example.

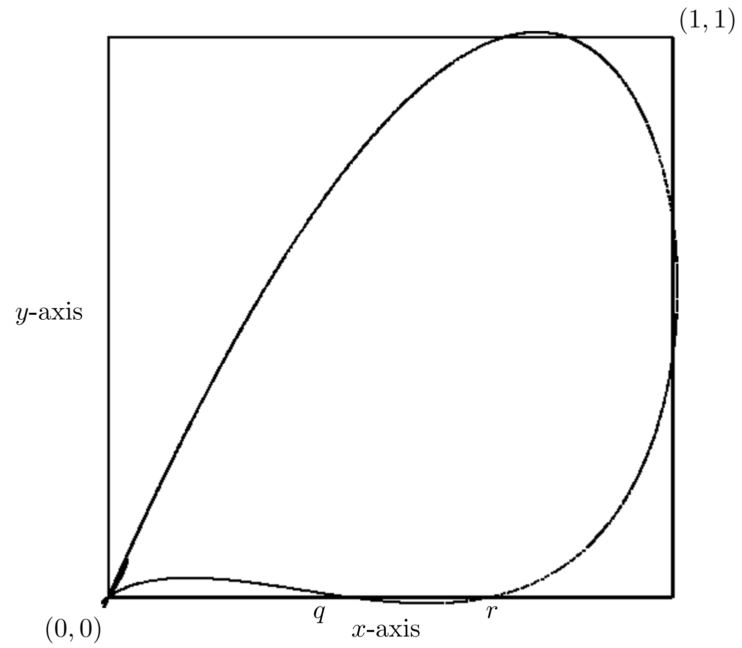

FIGURE 1. Unstable manifold of the origin for the delayed regulation map, $a = 2.28$. The unstable manifold intersects the stable manifold transversally at points q and r on the x-axis.

Example 3.1. A standard model of population is of the form

$$P_{n+1} = RP_n,$$

where P_n is the population at time n. According to Maynard Smith **Mathematical Ideas in Biology** [32], there are practical situations in which R depends on P_{n-1}. The reproduction rate for a herbivorous species may depend most strongly on the amount of vegetation eaten in the previous year. A version of this model is the delayed regulation model [3]:

$$P_{n+1} = aP_n(1 - P_{n-1}).$$

Making the change of variable $x_n = P_{n-1}$ and $y_n = P_n$, we get

$$\begin{aligned} x_{n+1} &= y_n \\ y_{n+1} &= a y_n (1 - x_n). \end{aligned}$$

This is clearly a noninvertible map, since the entire x-axis gets mapped immediately to the origin. Thus the fixed point at the origin has the x-axis as its local stable manifold. Just as for diffeomorphisms, we can define the global stable and unstable manifolds to be the set of points through which there is an orbit converging forward (resp. backward) to the fixed point under iteration. Note that this map has a unique orbit through every point, but there are many backwards orbits through a given point. In order for a point to be in the unstable manifold, it is only necessary for one of its backward orbits to converge to the fixed point. When the stable and unstable manifolds cross, the intersection is a homoclinic point, which is contained in an orbit with both forward and backward iterates converging to the fixed point. For more about stable and unstable manifolds and homoclinic points for noninvertible maps, see [29].

For the delayed regulation model at $a > 2.27$, the stable and unstable manifolds to the origin intersect transversally, as shown in Figure 1. Thus there are transverse homoclinic orbits to the fixed point at the origin.

The closure of such a homoclinic orbit for the delayed regulation map is a likely candidate for a hyperbolic set. However, for such an orbit, there are no invariant unstable directions. This arises from the fact that points on the x-axis map directly to the fixed point, rather than converging to it; thus in the two homoclinic orbits, there exist points $q = (q_1, 0)$ and $r = (q_2, 0)$ such that $f(q) = f(r) = (0, 0)$. Note that the unstable eigenspace of the origin is $\{(1, a)\}$. Thus to have invariance we would need vectors $v \in T_q R^2$ and $u \in T_r R^2$ such that $Df(q)v$ and $Df(r)u$ are multiples of $(1, a)$. However, since the derivative

$$Df(x, y) = \begin{pmatrix} 0 & 1 \\ -ay & a(1 - x) \end{pmatrix} \tag{1}$$

maps point q and r by $Df(q_i, 0)(\xi, 1) = (1, a(1 - q_i))$, there are no such vectors. This phenomenon was first described in work of Steinlein and Walther [34]. Figure 2 shows the global unstable manifold in a neighborhood of the origin. The three lines are candidates for $E^u_{(0,0)}$: the unstable eigenspace $(1, a)$ of the derivative at the origin, and $(1, a(1 - q_1))$ and $(1, a(1 - q_2))$, the images under the derivative of the tangent vector corresponding to the homoclinic points q and r on the x-axis.

Based on these ideas, Steinlein and Walther [33, 34] generalize the definition of hyperbolic sets to a definition for C^1 noninvertible maps on Banach spaces. For their definition, they assume that there exist stable subspaces mapping invariantly, and eventually contracting. They also assume that there exist unstable subspaces, not mapping invariantly, but with unstable projections eventually expanding. For this definition, they prove the shadowing lemma by making some adjustments to the traditional functional analytic techniques. The development here of the theory of hyperbolic sets follows a very different approach, by exploiting the structure of smooth relations.

4. **Illustration of hyperbolicity.** This section illustrates the concepts of hyperbolic sets for relations. First are a series of examples in the simple case of hyperbolic fixed points. The final example shows how these ideas apply to give a definition of hyperbolic sets.

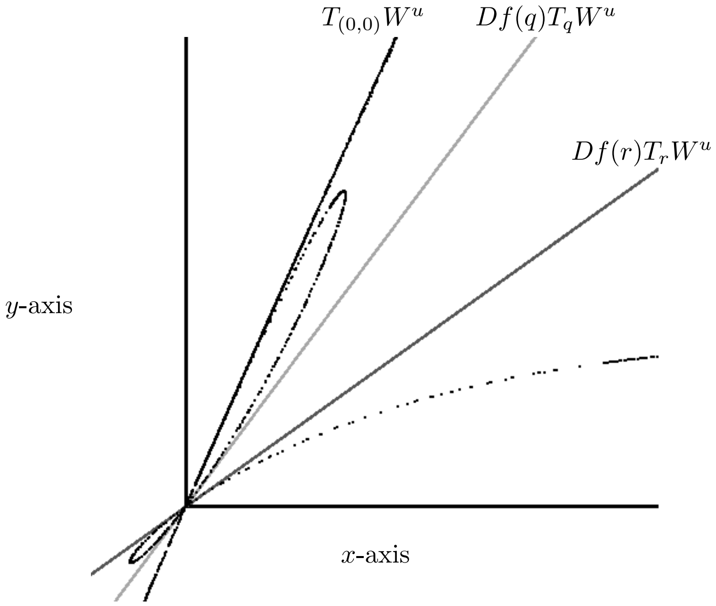

FIGURE 2. W^u near the origin for the delayed regulation model. The three tangents to W^u show that there is no invariantly mapping unstable subspace.

Example 4.1. Consider the graph of the following linear diffeomorphism f on R^2 with hyperbolic fixed point $(0,0)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{for } 0 < |a| < 1 < |b|. \tag{2}$$

Note the derivative at any hyperbolic fixed point with a stable and an unstable direction can be written in this form, using an appropriate change of coordinates.

We can solve for y in terms of x and y'. Thus the graph of f is the graph of a contraction ϕ_1 in *skewed coordinates*:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} x \\ y' \end{pmatrix}. \tag{3}$$

The k^{th} iterate of the original map is

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

Writing the graph of the k^{th} iterate in the same skewed coordinates, gives the following contraction ϕ_k. Note that this is found by looking at f^k and *not* by iterating ϕ_1, although in this case both methods give the same answer.

$$\begin{pmatrix} x_k \\ y \end{pmatrix} = \begin{pmatrix} a^k x \\ b^{-k} y_k \end{pmatrix}. \tag{5}$$

The advantage of the skew coordinate system is that it gives the graph of a map in terms of a contraction. For example, consider the limit of the ϕ_k; it exists and is equal to the map which is identically zero; explicitly, $\lim_{k\to\infty} \phi_k$ is the following map in skewed coordinates on R^2:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y' \end{pmatrix}. \tag{6}$$

Notice that $\lim_{k\to\infty} f^k$ is not well-defined. The limit of ϕ_k in the skewed coordinate system does not correspond to a function in the original coordinates. However, we can gain information about the limit behavior of the original map from the graph of this limit relation. Namely, the projection of the graph to the xy- plane is the x-axis, the stable manifold. The projection of the graph to the $x'y'$- plane is y'-axis, the unstable manifold.

Example 4.2. The trick in Example 4.1 still works if the linear map is noninvertible; e.g. if $a = 0$. The map becomes:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ by \end{pmatrix}, \qquad \text{for } 1 < |b|, \tag{7}$$

which can still be expressed in the same skewed coordinates as before:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{b}y' \end{pmatrix}. \tag{8}$$

The limit of the ϕ_k, the k^{th} iterate written in skewed coordinates, is the same as before. Indeed, the stable and unstable manifolds are once again the x-axis and y'-axis respectively.

Example 4.3. If we allow the stretching term b in Example 4.1 to increase without bound, the graph of the map of f converges to $\{(u, 0, au, v) : (u, v) \in R^2\}$. This is no longer the graph of a function from the xy-plane to the $x'y'$- plane. It is only a relation.

Definition 4.4 (Relation). *A relation on a space Z is a subset of $Z \times Z$. Viewing this in terms of iteration, an iterate of z under relation F is a point z' such that $(z, z') \in F$. Notice that iterates of a point are not necessarily unique; nor do iterates necessarily exist.*

The relation in this example is a two-dimensional plane which is a subset of R^4 with second coordinate always equal to 0. A point $(x, y) \in R^2$ has no iterates unless $y = 0$. A point $(x, 0)$ has as iterates every point of the form (ax, y'), $y' \in R$. Thus the origin is still a fixed point under iteration. Since points on the x-axis have k^{th} iterates of the form $(a^k x, 0)$, which converge to the origin, the x-axis is contained in (and in fact equal to) the stable manifold. Likewise, every point on the y-axis is an iterate of the origin. Thus the y-axis is contained in (and in fact equal to) the unstable manifold.

We can also use the technique in Examples 4.1 and 4.2 to see this; although there is no longer a map, the limit of b increasing without bound corresponds to $b = \infty$; i.e. $\frac{1}{b} = 0$. Thus although our example is no longer a map, it is the graph of a function in skewed coordinates:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} ax \\ 0 \end{pmatrix}. \tag{9}$$

In this case, as in Examples 4.1 and 4.2, the limit of the iterates as expressed in skewed coordinates exists and is equal to the zero function, and again the projections of the graph of this zero function are the stable and unstable manifolds.

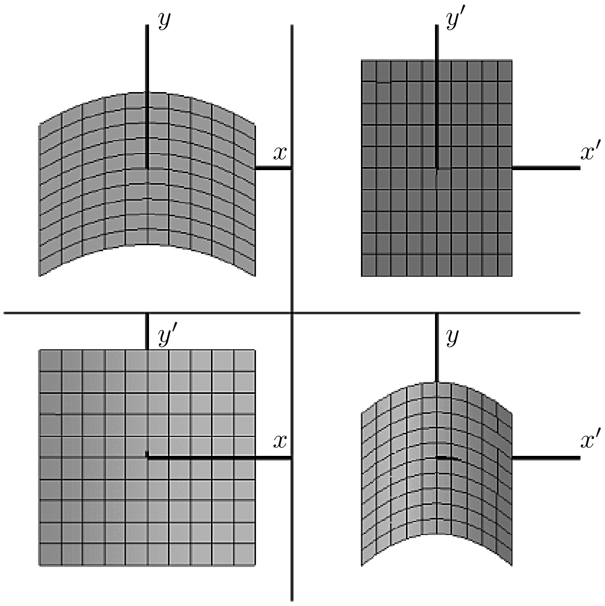

FIGURE 3. Projections of the graph of ϕ_1 resulting from the map in Example 4.5. Domain and range $[-.3, .3] \times [-.3, .3]$, $a = .7$, $b = 1.43$, and $c = 1$.

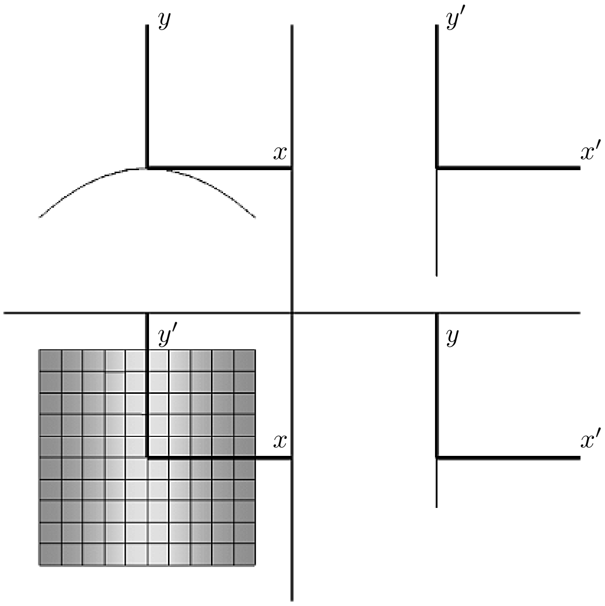

FIGURE 4. Same projections as in Figure 3, this time of the limit of the skewed functions $\lim \phi_k$.

Example 4.5. The implicit function theorem guarantees that for a hyperbolic fixed point of a nonlinear map, it is still possible to write the graph locally as the graph of a contraction in skew coordinates. Again in the nonlinear case, limits of the skew coordinate functions exist and give important information about the limit behavior of the original map. Here is a contrived quadratic example to illustrate this idea. Note that the map f on R^2 has a hyperbolic fixed point (0,0):

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax \\ b(y+cx^2) \end{pmatrix}, \qquad \text{for } 0 < a < 1 < b. \tag{10}$$

Since the axes are again the stable and unstable directions, we choose the axes for the skewed coordinate directions as before. The map represented in the skewed coordinate system gives the following function ϕ_1:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} ax \\ \frac{1}{b}y' - cx^2 \end{pmatrix}. \tag{11}$$

Note that ϕ_1 is a contraction in a sufficiently small neighborhood of the fixed point. Figure 3 shows the graph of ϕ_1 with domain $[-.3,.3]\times[-.3,.3]$ [15]. Since the graph of a map from R^2 to R^2 is in R^4, the figure consists of projections of the graph to coordinate planes. The projections have the following relationship to the maps f and ϕ_1: f maps the region in the xy- plane to the region in the $x'y'$- plane. ϕ_1 maps the region in the xy'- plane to the region in the $x'y$- plane.

The graph of f^k is the graph of ϕ_k in skew coordinates, which is once again a contraction close to the fixed point. The limit $\lim_{k\to\infty}\phi_k$ exists. It is given by:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{bc}{b-a^2}x^2 \end{pmatrix}. \tag{12}$$

Figure 4 is the graph of $\lim_{k\to\infty}\phi_k$. Once again, the projection to the xy-plane is the stable manifold, the projection to the $x'y'$-plane is the unstable manifold.

The discussion above for fixed points also applies to sets. Namely, using the adapted metric and writing everything in terms of stable and unstable subspaces, the derivative at a point in a hyperbolic set for a diffeomorphism has the same form as the derivative of a hyperbolic fixed point. It can therefore be written as the graph of a contraction in terms of skew coordinates. Further, if we assume only that there is some set of skew coordinates in which the derivative is a graph of a contraction, and do not require the contraction to be of a special form, we remove the assumption of invariance of stable and unstable subspaces.

Example 4.6. For the delayed regulation map f, consider the homoclinic orbit $\{q_{-k}\}$ through $q = q_o$. (This argument will work for the orbit through r as well.) For points in the orbit in a neighborhood of the origin, let $E^u_x = T_xW^u(0)$ and $E^s_x = T_xW^s(0)$. Recursively define $E^u_{q_{-k}} = Df(q_{-k-1})E^u_{q_{-k-1}}$. For E^s, let $E^s_{q_o}$ be the x-axis, since this is the subspace mapping to E^s_x under the derivative. The map is invertible except on the x-axis, so for $k > 0$ we can also let $E^s_{q_{-k}} = Df(q_{-k-1})E^s_{q_{-k-1}}$. Since the unstable manifold is transverse to the stable manifold at each homoclinic point on the x-axis, the same must be true for all points in the homoclinic orbit. In addition, at q, the map is an isomorphism on E^u.

From the above, it is possible to find a new continuous metric such that Df is contracting on stable subspaces and expanding on unstable subspaces. See [28] for details. In fact, the graph of the derivative $Df(q_{-k}) : E^s_{q_{-k}} \times E^u_{q_{-k}} \to E^s_{q_{-k+1}} \times E^u_{q_{-k+1}}$ is the graph of a linear contraction $D\phi^{q_{-k}} : E^s_{q_{-k}} \times E^u_{q_{-k+1}} \to E^s_{q_{-k+1}} \times E^u_{q_{-k}}$.

Looking at the limits of skew coordinate contractions gives information about the limit behavior of the map. Projections of the limit maps give stable and unstable manifolds for points in the hyperbolic set. See Section 8. In addition, the contraction on the tangent space to each point gives rise to a local contraction in sequence space. This contraction leads quite naturally to a proof of the shadowing lemma. See Section 7.

As mentioned in the discussion before the final example, in order to remove the assumption that the stable and unstable subspaces map invariantly, we need to assume that hyperbolicity only guarantees the existence of a skewed coordinate linear contraction, not a diagonal matrix; in this example, the matrix corresponding to $D\phi^{q_o} : E^s_{q_o} \times E^u_0 \to E^s_0 \times E^u_{q_o}$ is *not* diagonal. This is exactly because the subspaces do not map invariantly.

5. **Definitions.** This section develops the definition of hyperbolicity for relations. Recall that relations on Z were defined as subsets of $Z \times Z$ and were viewed in terms of iteration. Here are some definitions in this context. We denote z having an iterate z' under relation F by $z \overset{F}{\mapsto} z'$. For example, z_o is a fixed point exactly when $z_o \overset{F}{\mapsto} z_o$. The following basic definitions for relations follow the discussion in [22].

Notation: In this paper, capital letters F, G, and H represent relations, whereas a lower case f, g, and h represent functions.

Definition 5.1 (Composition for relations)**.** *Given relations G and H on set Z, $H \circ G$ is the relation given by*

$$\{(z, z'') : \exists\ z' \in Z, (z, z') \in G \textit{ and } (z', z'') \in H\} \tag{13}$$

Notation: If I is an interval of integers and $z_k \in Z$ for all $k \in I$ is a sequence of points in Z, then we denote

$$\{z_k\}_{k \in I} = \left\{ \begin{array}{c} (z_i, z_{i+1}, ..., z_j), \text{ if } I = [i, j] \\ (..., z_{j-1}, z_j), \text{ if } I = (-\infty, j] \\ (z_i, z_{i+1}, ...), \text{ if } I = [i, \infty) \end{array} \right\} \tag{14}$$

Definition 5.2 (Orbits for relations)**.** *Given relation F on space Z, an orbit through z is a sequence $\{z_k\}_{k \in I}$ such that $z = z_i$ for some $i \in I$, and $(z_k, z_{k+1}) \in F$ whenever $k, k+1 \in I$. If $I = [i, \infty)$ then $\{z_k\}$ is called an infinite forward orbit. If $I = (-\infty, i]$ then $\{z_k\}$ is called an infinite backward orbit.*

Definition 5.3 (C^r relations)**.** *If F is a relation on a smooth manifold Z, then F is C^r when it is a C^r embedded submanifold of $Z \times Z$.*

Definition 5.4 (Tangent relation)**.** *Given a smooth relation F on R^p, the tangent relation TF on R^{2p} is the tangent bundle of F.*

Based on the motivation from the previous section, we now give the definition of hyperbolic sets for smooth relations.

Definition 5.5 (Splitting for R^n)**.** *A splitting for R^n is a pair of subspaces (E^s, E^u) satisfying*
$R^n = E^s \times E^u$.

Definition 5.6 (4-Splitting for $R^n \times R^n$)**.** *A 4-splitting for $R^n \times R^n$ is a pair of splittings for R^n, (E^s_1, E^u_1) and (E^s_2, E^u_2), satisfying dim(E^s_1)=dim(E^s_2) and dim(E^u_1) = dim(E^u_2).*

Definition 5.7 (Hyperbolic linear relation with respect to a 4-splitting)**.** *If A is an n- dimensional linear relation on R^n, then A is hyperbolic with respect to the 4-splitting $(E^s_1, E^u_1, E^s_2, E^u_2)$ if in terms of the max norm on the subspaces of the* *splitting, there is a linear contracting map* $M : E_1^s \times E_2^u \to E_2^s \times E_1^u$, $|M| < \lambda < 1$ *such that* A *is the graph of* M*. Thus*

$$A = \left\{ (x, y, x', y') : \begin{pmatrix} x' \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y' \end{pmatrix} \right\}.$$

We say A *has a contraction constant* $|M|$ *with respect to this 4-splitting.*

Definition 5.8 (Compact hyperbolic sets for C^1 relations)**.** *Let* F *be a* C^1 *relation on* R^n*. Let* K *be a compact set in* R^n*.* K *is a hyperbolic set for* F *when the following conditions hold:*

There is a continuous splitting of the tangent space of R^n *over* K*. Denote it:* $T_z R^n = E_z^s \times E_z^u$*.*

There is a constant $\lambda < 1$*.*

There is some metric on the subspaces of the splitting in a neighborhood of K *such that at every* $(z, w) \in F \cap (K \times K)$, $T_{(z,w)}F$ *is a hyperbolic linear relation with contraction constant* $< \lambda$ *with respect to the 4-splitting* $(E_z^s, E_z^u, E_w^s, E_w^u)$*.*

The following lemma relating hyperbolicity and Lipschitz relations is what we actually use in the proofs of the theorems in the next section.

Lemma 5.9 (Hyperbolic implies Lipschitz)**.** *If* $(z, w) \in F \cap (K \times K)$*, with* F *and* K *as above, then in a neighborhood of* (z, w)*,* F *is the graph of a* Lip_λ *function* $f : E_z^s \times E_w^u \to E_w^s \times E_z^u$*.*

Proof. By the Implicit Function Theorem. □

6. **Cones.** Although hyperbolic sets for diffeomorphisms always have invariant subspaces, it is often difficult to find them. To avoid this problem, hyperbolicity is often formulated in terms of an equivalent condition on stable and unstable cones. We show below that a hyperbolic set for a relation satisfies a generalization of the cone condition.

For diffeomorphisms, the cone formulation says that on some compact invariant set K, if the derivative is expanding on a cone of vectors roughly in the unstable direction and contracting on a cone of vectors roughly in the stable direction, then K is a hyperbolic set for the diffeomorphism [26]. The following makes this more precise.

Definition 6.1 (Stable and unstable cones)**.** *Given* $\alpha > 0$*, and a splitting of the tangent space at each point* $T_z R^n = E_z^s \times E_z^u$*, then the stable and unstable* α *cones are defined by:*

$$C_z^s = \{(v_s, v_u) \in E_z^s \times E_z^u : |v_u| \leq \alpha |v_s|\}$$

$$C_z^u = \{(v_s, v_u) \in E_z^s \times E_z^u : |v_s| \leq \alpha |v_u|\}.$$

Definition 6.2 (Cone condition for diffeomorphisms)**.** *If* f *is a diffeomorphism and* K *is a compact invariant set, then* K *satisfies the cone condition if there is a continuous metric and a continuous splitting of* K *such that* $Df(z)C_z^s \subset C_{f(z)}^s$*,* $Df^{-1}(f(z))C_{f(z)}^u \subset C_z^u$*, and* Df *is uniformly contracting on the stable cone and uniformly expanding on the unstable cone.*

Theorem 6.3 (Hyperbolic sets and cones for diffeomorphisms)**.** *If* f *is a diffeomorphism and* K *is a compact invariant set, then* K *satisfies the cone condition if and only if* K *is a hyperbolic set.*

This theorem is due to Newhouse and Palis [26].

The new definition of hyperbolic sets for relations is similar to this cone formulation. Here is a natural generalization of the cone condition to relations, and a statement that hyperbolic sets for relations satisfy this cone condition.

Definition 6.4 (Cone condition for relations). *Let F be a smooth relation and K a compact set. Then F satisfies the cone condition on K if there is some continuous splitting and a continuous metric, and a uniform $\lambda < 1$ such that for all $(z, w) \in F \cap (K \times K)$, vectors in the unstable λ cone at z relate only to vectors in the unstable λ cone at w under $T_{(z,w)}F$. Further, it is backward λ-contracting. In other words, if a vector v in the unstable cone relates to a vector v', then $|v| < \lambda|v'|$.*

Similarly, vectors in the stable λ cone at w only come from the stable λ cone at z under $T_{(z,w)}F$, and are λ-contracting.

Lemma 6.5 (Equivalence of cones and hyperbolic sets for relations). *Assume that F is a relation and K is a compact set. Then K is a hyperbolic set for F if and only if F satisfies the cone condition on K.*

Proof. $(\Rightarrow)$

Let K be a hyperbolic set for F. Choose a nonzero vector (v_s, v_u) in the unstable cone at z. Suppose (v_s, v_u) relates to a vector (v_s', v_u') at w under the derivative relation. Then

$$|(v_s', v_u)| < \lambda|(v_s, v_u')|.$$

By hypothesis, $|v_s| < \lambda|v_u|$, so $|v_u'| > |v_s|$. Thus $|v_s'| < \lambda|v_u'|$, so the vector is in the unstable cone. Furthermore the derivative is expanding vectors, since $|(v_s, v_u)| = |v_u| < \lambda|v_u'| = \lambda|(v_s', v_u')|$. A similar proof holds for the stable cones.

$(\Leftarrow)$

Let F on K satisfy the cone condition. Let $(x, y) \in (K \times K) \cap F$. Writing vectors in terms of the splitting for the cones, let $\{(u_{1i}, u_{2i}, v_{1i}, v_{2i})\}_{i=1}^n$ be a basis for $T_{(x,y)}F$. Then $\{(u_{1i}, v_{2i})\}_{i=1}^n$ must be linearly independent, because if not, there is some vector $\{(0, u_{2i}, v_{1i}, 0)\} \in T_{(x,y)}F$, which implies that a vector in C^u relates to a vector in C^s, which violates the cone condition. Thus $T_{(x,y)}F$ is a contraction in terms of the splitting given by the coordinates for the cones. This is a linear contraction by the cone condition and linearity. □

7. **Shadowing.** The shadowing lemma states that near a hyperbolic set, making small errors on each iteration still gives a reasonable picture of the dynamics. The shadowing lemma for diffeomorphisms is due to Bowen [5]. Here we give a proof of it for hyperbolic sets for smooth relations. The proof is functional analytic in nature. However, it seems to be conceptually simpler than the standard functional analytic proofs for diffeomorphisms, as the definition of hyperbolic sets in terms of a contraction allows the straightforward application of the contraction mapping theorem.

Notation: For $\gamma > 0$ and set S, the closed ball of radius γ is denoted by $B_\gamma(S)$.

Definition 7.1 (Pseudo-orbit). *A sequence $\{z_i\}_{i \in I}$ is called a δ-pseudo-orbit for the relation F as long as $dist((z_i, z_{i+1}), F) < \delta$ whenever $i, i+1 \in I$. More elegantly in terms of relations, a δ-pseudo-orbit for F is an orbit of $B_\delta(F)$.*

Definition 7.2 (Shadow). *An orbit of the relation F $\{w_i\}_{i \in I}$ is said to ϵ-shadow a sequence $\{y_i\}_{i \in I}$ if for all $i \in I$, $dist(w_i, y_i) < \epsilon$.*

If in some set, every sufficiently small pseudo-orbit has a unique nearby shadow, the set is said to have the *shadowing property.* The following lemma says that hyperbolic sets have the shadowing property.

Theorem 7.3 (Shadowing)**.** *If K is a hyperbolic set for F, then for any $\epsilon > 0$, there is a $\delta > 0$ such that any δ-pseudo-orbit in $B_\delta(K)$ is ϵ-shadowed by an orbit of F. If ϵ is small enough and the pseudo-orbit is bi-infinite, then its ϵ-shadow is unique. Further, if the pseudo-orbit is periodic, so is its shadow.*

The proof is conceptually simple. In summary, uniformly close to a pseudo-orbit near K, F is the graph of a contraction. Associated with F, there is a relation on the space of sequences. Near the pseudo-orbit, this sequence relation is the graph of a contraction from the space of sequences to itself. By the contraction mapping theorem, this contraction has a unique fixed point, which is the shadowing orbit we wanted to find. Here are the details of the argument.

Proof. Assume we are given relation F and hyperbolic set K.

Given ϵ sufficiently small, there is a small enough δ so that if $z, w \in B_\epsilon(K)$ and $(z, w) \in B_\delta(F)$, then F is the graph of a Lipschitz contraction on $B_\epsilon(z, w)$.

Assume that $z^* = \{z_i\}_{i \in I}$ is a δ-pseudo-orbit in $B_\delta(K)$.

For sequence z^*, note that at each point there is an induced splitting, defined by $E^{s*}_{z^*} = \left\{E^s_{z_i}\right\}$ and $E^{u*}_{z^*} = \left\{E^u_{z_i}\right\}$. Thus there is an induced splitting for any sequence in $B_\epsilon(z^*)$.

For sequence $x^* \in E^{s*}_{z^*}$, define the norm $\|x^*\| = \sup_i |x_i|$.

Likewise, for $y^* \in E^{u*}_{z^*}$, let $\|y^*\| = \sup_i |y_i|$.

On $(x^*, y^*) \in E^{s*}_{z^*} \times E^{u*}_{z^*}$, use the norm $\|(x^*, y^*)\| = \max(\|x^*\|, \|y^*\|)$.

Let F^* be the relation induced by F. Namely, on the space of bi-infinite sequences in $B_\epsilon(z^*)$:

$w^* \overset{F^*}{\mapsto} \rho^*$ if and only if for all i, $w_i \overset{F}{\mapsto} \rho_{i+1}$

Lemma 7.4. *F^* is the graph of a Lip_λ function f^* induced by f.*

Proof. Assume $(x^*, y^*) \overset{F^*}{\mapsto} (\alpha^*, \beta^*)$. Then $(x_i, y_i) \overset{F}{\mapsto} (\alpha_{i+1}, \beta_{i+1})$, which implies $f(x_i, \beta_{i+1}) = (\alpha_{i+1}, y_i)$. Thus there is a function f^* induced on sequences such that $f^*(x^*, \beta^*) = (\alpha^*, y^*)$.

To show that this function is $\mathrm{Lip}_{\lambda'}$, assume $(\xi^*, \eta^*) \overset{F^*}{\mapsto} (\theta^*, \mu^*)$. Thus $(\xi_i, \eta_i) \overset{F}{\mapsto} (\theta_{i+1}, \mu_{i+1})$. By assumption, we know $\max(|\alpha_{i+1} - \theta_{i+1}|, |y_i - \eta_i|) \leq \lambda \max(|x_i - \xi_i|, |\beta_{i+1} - \mu_{i+1}|)$. Taking the sup of both sides, we see that $\max(\|\alpha^* - \theta^*\|, \|y^* - \eta^*\|) \leq \lambda \max(\|x^* - \xi^*\|, \|\beta^* - \mu^*\|)$. Thus f^* is Lip_λ. □

Lemma 7.5. *F^* has a fixed point if and only if f^* has a fixed point.*

Proof. Follows from the fact that $((x^*, y^*), (\alpha^*, \beta^*)) \in F^*$ if and only if $f^*(x^*, \beta^*) = (\alpha^*, y^*)$. □

Lemma 7.6. *By the contraction mapping theorem, f^* has a unique fixed point.*

Proof. The space of bi-infinite sequences on compact balls of radius ϵ along with the norm previously described is a Banach space. Thus we can use the contraction mapping theorem. □

From the above lemma, F^* has a unique fixed point. A fixed point of F^* is an orbit of F. Thus this fixed point sequence is the unique shadow of the pseudo-orbit z^*.

This completes the proof of the shadowing lemma. □

8. **Stable manifold theorem for hyperbolic sets of relations.** This section describes a generalization of the stable manifold theorem for hyperbolic sets which holds for noninvertible maps and relations.

The stable manifold theorem for hyperbolic sets of diffeomorphisms says that for every point in a hyperbolic set, the set of points with nearby forward orbits and the set of points with nearby backward orbits both form smooth manifolds. It was originally stated by Hirsch and Pugh [14]. In the case of relations, the nonuniqueness of orbits implies that this statement breaks down.

The delayed regulation model previously discussed is an example of a hyperbolic set for a noninvertible map such that there are multiple unstable manifolds to a point. At $a = 2.28$ the transverse homoclinic orbits form hyperbolic sets. However, within the hyperbolic set, the unstable manifold to the origin is not unique. By the same calculation which showed there was no invariant unstable direction at the origin, there are many curves which comprise the unstable manifold to the origin, as can be seen in Figure 2. Each of these curves follows one of the backward orbits to the origin.

As the delayed regulation example suggests, the stable manifold theorem for hyperbolic sets for relations is a theorem about orbits rather than about points. Namely, it shows that for each orbit in a hyperbolic set, there are stable and unstable manifolds, consisting of points with locally unique forward and backward orbits respectively. Since for a given point, orbits of a noninvertible map or relation are neither guaranteed to exist nor to be locally unique, this is a powerful result.

Definition 8.1 (Local manifolds for orbits). *Let $\{z_k\}_{k\in I}$ be an orbit for a smooth relation F on space Z. The stable (unstable) manifold to $\{z_k\}$ at z_i exists as long as $[i,\infty) \subset I$ (respectively $(-\infty, i] \subset I$). These manifolds are denoted $W^s_{\{z_k\}}(z_i)$ and $W^u_{\{z_k\}}(z_i)$, and are defined as follows:*

$W^s_{\{z_k\}}(z_i) = \{w \in Z :$ *there exists an infinite forward orbit* $\{w_k\}$ *through* w *such that* $w_k \to z_k$ *as* $k \to \infty\}$.

$W^u_{\{z_k\}}(z_i) = \{w \in Z :$ *there exists an infinite backward orbit* $\{w_k\}$ *through* w *such that* $w_k \to z_k$ *as* $k \to -\infty\}$.

The local stable and unstable manifolds consist of points in the stable and unstable manifolds whose iterates always remain close to the original orbit. Specifically, for $\epsilon > 0$, $W^s_{\{z_k\}}(z_i, \epsilon)$ and $W^u_{\{z_k\}}(z_i, \epsilon)$ are defined exactly as above, except in each case, we must have the additional condition that for all $k \in I$, $w_k \in B_\epsilon(z_k)$.

Theorem 8.2 (Stable manifolds for relations). *Let F be a C^k smooth relation with compact hyperbolic set K. Assume $\{z_k\}_{k\in I}$ is an orbit of F contained in K.*

For sufficiently small ϵ, if $[i,\infty) \subset I$, the local stable manifold $W^s_{\{z_k\}}(z_i, \epsilon)$ is the graph of a C^k smooth function $E^s_{z_i} \to E^u_{z_i}$. Further, each point q_o in the local stable manifold has a locally unique forward orbit. In other words, there is only one forward orbit through q_o converging to the orbit $\{z_k\}$.

Similarly, for the unstable case, if $(-\infty, i] \subset I$, and ϵ is sufficiently small, then the local unstable manifold $W^u_{\{z_k\}}(z_i, \epsilon)$ is the graph of a C^k smooth function $E^u_{z_i} \to E^s_{z_i}$. Further, each point r_o in the local unstable manifold has a locally unique backward orbit.

Before beginning the proof, we develop some notation and reformulate and prove the Lipschitz version of the theorem in terms of this notation. Then we prove the smooth version of the theorem using a proof similar to the graph transform method. Assume F is a relation with hyperbolic set K, and $\{z_k\}_{k\in I}$ is an orbit in K.

For $(z, w) \in R^n \times R^n$, denote the projections to the two coordinates by π_1 and π_2; namely, $\pi_1(z, w) = z$ and $\pi_2(z, w) = w$.

Definition 8.3 (Forward shadow). *S_i^1 is the set of pairs ϵ-shadowing forward for one iterate. i.e.*

$$S_i^1 = \{(u, v) \in F : (u, v) \in B_\epsilon(z_i, z_{i+1})\}.$$

Equivalently, $S_i^1 = F \cap B_\epsilon(z_i, z_{i+1})$.

S_i^k is the set of pairs which are connected by a k-forward shadow. Equivalently, in terms of composition of relations,

$$S_i^k = S_{i+k-1}^1 \circ \cdots \circ S_{i+1}^1 \circ S_i^1.$$

Likewise, for backward shadowing,

Definition 8.4 (Backward shadow). *S_i^{-1} is the set of pairs ϵ-shadowing backward for one iterate. Thus, $S_i^{-1} = F \cap B_\epsilon(z_{i-1}, z_i)$.*

S_i^{-k} is the set of pairs which are connected by a k-backward ϵ-shadow. In terms of composition of relations,

$$S_i^{-k} = S_i^{-1} \circ S_{i-1}^{-1} \circ \cdots \circ S_{i-k+1}^{-1}.$$

By definition $W^s(z_i) = \lim_{k\to\infty} \pi_1(S_i^k)$, and $W^u(z_i) = \lim_{k\to\infty} \pi_2(S_i^{-k})$. Thus the following lemma in terms of the new notation is equivalent to the Lipschitz version of the stable manifold theorem for hyperbolic sets for relations.

Lemma 8.5. *Under the hypotheses of Theorem 8.2, there is a constant $0 < \lambda < 1$ such that $\lim_{k\to\infty} \pi_1(S_i^k)$ exists and is the graph of a Lip$_\lambda$ function $E_{z_i}^s \to E_{z_i}^u$. Each point in this limit set has a unique forward orbit converging to $\{z_k\}$. Also $\lim_{k\to\infty} \pi_2(S_i^{-k})$ exists and is the graph of a Lip$_\lambda$ function $E_{z_i}^u \to E_{z_i}^s$. Each point in this limit set has a unique backward orbit converging to $\{z_k\}$.*

Proof. Here is the proof of the above lemma for the backward orbits case. Forward orbits case follows by symmetry.

Step 1. $\pi_2\left(S_i^{-k}\right) \subset \pi_2\left(S_i^{-(k-1)}\right)$.

This is because the set of points shadowing backward k iterates automatically must shadow backward for $k-1$ iterates.

Step 2. $\pi_2\left(S_i^{-k}\right)$ *is compact.*

This set is a projection of S_i^{-k}, which is compact since it is the composition of compact relations.

Step 3. *If δ is sufficiently small, then for every k and every $y \in E_{z_i}^u$, there exists x so $(x, y) \in \pi_2\left(S_i^{-k}\right)$.*

This follows from the following lemma on the composition of relations.

Lemma 8.6. *The composition of relations which are graphs of Lipschitz functions is a graph of a Lipschitz function. Specifically, if F is the graph of Lip$_\lambda$ function f, G the graph of Lip$_\lambda$ function g,*

$f : E_1^s \times E_2^u \to E_2^s \times E_1^u$, and

$g : E_2^s \times E_3^u \to E_3^s \times E_2^u$

then $G \circ F$ is the graph of a Lip$_\lambda$ function $E_1^s \times E_3^u \to E_3^s \times E_1^u$

Proof. **of the lemma:** By the Lipschitz Implicit Function Theorem. □

Step 4. $\lim_{k\to\infty} \pi_2\left(S_i^{-k}\right)$ *is the graph of a function from* $E^u_{z_i}$ *to* $E^s_{z_i}$.

From step 3, $\pi_2\left(S_i^{-k}\right)$ is nonempty. Thus $\lim_{k\to\infty} \pi_2\left(S_i^{-k}\right) = \cap_{k>0}\pi_2\left(S_i^{-k}\right)$ is nonempty since it is the intersection of a nested sequence of compact sets. Further, using the same reasoning, for every $y \in E^u_{z_i}$, there is an $x \in E^s_{z_i}$ such that $(x, y) \in \lim_{k\to\infty} \pi_2\left(S_i^{-k}\right)$.

Here is a useful estimate:

Lemma 8.7. *Assume* $\{w_{-k}\}$ *and* $\{u_{-k}\}$ *are backward orbits such that for some sufficiently small* $\gamma > 0$, *and* $i \le k < N$, *dist*(w_{-k}, z_{-k}) *and dist*$(u_{-k}, z_{-k}) < \gamma$. *Then by continuity of the splitting we can rewrite* $w_{-k} = (x_{-k}, y_{-k}), u_{-k} = (\xi_{-k}, \eta_{-k}) \in E^s_{z_{-k}} \times E^u_{z_{-k}}$. *Then*

$$\max(|x_i - \xi_i|, |y_{i-1} - \eta_{i-1}|) \le \max(\lambda^k |x_{i-k} - \xi_{i-k}|, \lambda|y_i - \eta_i|).$$

Proof. Since F is Lip$_\lambda$, $\max(|y_{i-1}-\eta_{i-1}|, |x_i-\xi_i|) \le \max(\lambda|x_{i-1}-\xi_{i-1}|, \lambda|y_i-\eta_i|)$. Call this right hand quantity Δ.

Similarly, $|x_{i-1} - \xi_{i-1}| \le \max(\lambda|x_{i-2} - \xi_{i-2}|, \lambda|y_{i-1} - \eta_{i-1}|)$.

Thus

$$|x_i - \xi_i| \le \Delta \le \max(\lambda^2|x_{i-2} - \xi_{i-2}|, \lambda|y_i - \eta_i|).$$

Continuing inductively gives the desired inequality. □

Corollary 8.8. *If* (x_i, y_i) *and* (ξ_i, η_i) *are both in* $\lim_{k\to\infty} \pi_2\left(S_i^{-k}\right)$, *then* $|x_i-\xi_i| \le \lambda|y_i - \eta_i|$.

Proof. Choose N so that $\lambda^N\epsilon \le \lambda|y_i-\eta_i|$. By assumption, there is some (x_{i-N}, y_{i-N}) and (ξ_{i-N}, η_{i-N}) such that $(x_{i-N}, y_{i-N}, x_i, y_i)$ and $(\xi_{i-N}, \eta_{i-N}, \xi_i, \eta_i) \in S_i^{-N}$.

From the preceding lemma,

$$\max(|x_i - \xi_i|, |y_{i-1} - \eta_{i-1}|) \le \max(\lambda^N|x_{i-N} - \xi_{i-N}|, \lambda|y_i - \eta_i|) \le \lambda|y_i - \eta_i|.$$

□

Step 5. $\lim_{k\to\infty} \pi_2\left(S_i^{-k}\right)$ *is the graph of a Lip*$_\lambda$ *function. Every point of the limit set has a unique backward orbit converging to the original orbit.*

That the limit set is the graph of a function follows from the previous step. That the function is Lipschitz follows from the above corollary.

To show that points have unique backward orbits in the limit set, consider the behavior of the inverse relation when restricted to the limit set. From the last line of the proof of the above corollary, the inverse relation is a contraction. A relation which is a contraction is a function. Thus the inverse relation when restricted to the limit set is actually a function. From this it is clear that points in the limit set have unique backward orbits and that the orbits converge to $\{z_k\}$. This completes the proof of the Lipschitz portion of the stable manifold theorem.

We now prove smoothness of stable and unstable manifolds, following a method similar to the graph transform method in [14] and the proof of the cone condition in [26]. Basically, we look at the image under the derivative TF of the graph of a smooth function from unstable to stable space, and a linear subspace in the unstable cone at each point. We show that this has an attracting fixed point in terms of the entire sequence. This fixed point turns out to be the unstable manifold and its derivative, showing W^u is C^1. Induction shows that it is also C^r.

In $B_\epsilon(z_{-i})$, choose a Lip$_\lambda$ function $\rho : E^u_{z_{-i}} \to E^s_{z_{-i}}$. Refer to the graphs of such ρ as unstable disks. Let $G(\rho)$ be the function $\rho' : E^u_{z_{-i+1}} \to E^s_{z_{-i+1}}$ such that graph(ρ') is the image of graph(ρ) under the relation F. Hyperbolicity guarantees that ρ' is an unstable disk.

In the tangent space, choose subspaces to each point $L : E^u_{z_{-i}} \to \mathrm{Lin}(TE^u, TE^s)$ in the unstable cone. Let $TG(\rho, L)$ be the function (ρ', L') which has as its graph the image of graph$(\rho, L_{gr(\rho)})$ under TF. One can show that $L' : E^u_{z_{-i+1}} \to \mathrm{Lin}(TE^u_{z_{-i+1}}, TE^s_{z_{-i+1}})$ is well defined, and by the cone condition, $L'(y')$ is always in the unstable cone.

Now look at G on sequences. Namely, let $(\{\rho_i, L_i\})$ be a sequence of functions defined for the entire backward orbit, denoted (ρ^*, L^*). Let $G^*(\rho^*) = (\rho^{*1})$ and let $TG^*(\rho^*, L^*) = (\rho^{*1}, L^{*1})$. All sequences converge uniformly to the sequence of unstable manifolds under G^* by the Lipschitz portion of the proof. For the sequence L^*, let $|L^*| = \sup_i \sup_y \|L_i(y)\|$, where $\|.\|$ denotes the linear norm. The following lemma says that fixing the base sequence, the fibers converge to a fixed point.

Lemma 8.9. *Fibers converge under TG^*. Specifically, if we fix the sequence ρ^*, then for m sufficiently large, and $TG^{*m}(\rho^*, L^*) = (\rho^{*m}, L^{*m})$ and $TG^{*m}(\rho^*, M^*) = (\rho^{*m}, L^{*m})$, then $|L^{*m} - M^{*m}| < C\lambda^m$.*

Proof. First note that $|L^{*m}|, |M^{*m}| < \lambda$, since unstable cones map to unstable cones. Choose (x, y, x^m, y^m) in graph$(\rho_{-i}, \rho^m_{-i+m})$. Choose $v^m \in TE^u_{z_{-i+m}}$ with norm one, and choose u and v so that

$$(L_{-i}(y)v, v, L^m_{-i+m}(y^m)v^m, v^m),$$

and

$$(M_{-i}(y)u, u, M^m_{-i+m}(y^m)v^m, v^m) \in TF_{(x,y,x^m,y^m)}.$$

Since TF expands vectors in the unstable cone, $1 = |v^m| > |u|, |v|$.

Using Lemma 8.7,

$$\begin{aligned} \|L^m_{-i+m}(y^m) - M^m_{-i+m}(y^m)\| &< \lambda^m |L_i(y)v - M_i(y)u| && (15)\\ &< \lambda^m (\|L_i(y)\||v| + \|M_i(y)\||u|), && (16)\\ &< \lambda^m (2\lambda). && (17) \end{aligned}$$

Since the right hand side is independent of y and i, $|L^{*m} - M^{*m}| < (2\lambda)\lambda^m$. □

From the previous lemma and the fact that TF is smooth, TG^* has an attracting fixed point. Since TG^* preserves derivatives, we now see that W^u is C^1. The proof of the C^k case proceeds by induction, analogously to the standard diffeomorphism proof. Namely, we assume that for all smooth relations H with hyperbolic structure, sequences of smooth unstable disks converge in C^{k-1} to the unstable manifold. Assume ρ^* is a sequence of C^k unstable disks for the hyperbolic splitting with respect to F. Then $(\rho^*, D\rho^*)$ are C^{k-1} unstable disks for the hyperbolic splitting with respect to TF. By induction, this must converge in C^{k-1}. Thus ρ^* converges in C^k to the sequence of unstable manifolds. This completes the proof. □

8.1. **Related remarks.** Since the definition of hyperbolicity is equivalent to the cone condition, robustness of hyperbolic sets follows immediately:

Theorem 8.10 (Robustness)**.** *If K is a compact hyperbolic set for relation F, and G is a relation C^1 close to F, then K is a hyperbolic set for G.*

In addition, from the proof of the stable manifold theorem, the manifolds vary continuously with the orbit:

Theorem 8.11 (Continuous change of stable and unstable manifolds)**.** *The unstable (stable) manifolds to an orbit of a relation vary continuously as the orbit varies in the inverse (forward) limit norm.*

9. **Examples of hyperbolic sets.**

Example 9.1. A hyperbolic fixed point z_o of a smooth relation F is defined in [22] as a fixed point such that in terms of a splitting $E^s \times E^u$ of the tangent space $DF(z_o)$ is of the form:

$$\left\{ \begin{pmatrix} x \\ By' \\ Ax \\ y' \end{pmatrix} : x \in E^s, y' \in E^u \right\}, \tag{18}$$

where A and B are matrices, and $|A|, |B| < 1$. Clearly a hyperbolic fixed point for a smooth relation is a hyperbolic set.

Example 9.2. For diffeomorphisms, the definition for relations is equivalent to the traditional definition. In other words, we have the following theorem;

Theorem 9.3. *If $f : R^n \to R^n$ is a diffeomorphism, and K is a compact invariant set under f, then K is a hyperbolic set for f by the traditional definition if and only if K is a hyperbolic set for the relation graph(f) under the relations definition.*

Proof. By the Mather adapted norm, the diffeomorphism conditions imply that the new definition holds. The cone condition implies that the converse holds as well. □

Example 9.4 (Homoclinic orbits). As outlined in Example 4.6, the homoclinic orbits of the delayed regulation map are hyperbolic sets. Thus shadowing holds, and it is possible to show the existence of recurrent behavior near the orbit.

For diffeomorphisms, all transverse homoclinic orbits are embedded in hyperbolic sets. However, this is not always true for noninvertible maps and relations. There are examples of noninvertible maps and relations with transverse homoclinic orbits, but for which the shadowing lemma does not hold [28]. Since the shadowing lemma holds for all hyperbolic sets, these examples cannot ever be embedded in hyperbolic sets. There are sufficient conditions for homoclinic orbits for noninvertible maps and relations to be hyperbolic sets [28, 34, 33].

Example 9.5 (Snap-back repellers). A special case of a homoclinic orbit which is always a hyperbolic set is *snap-back repellers* [20], defined as follows.

Definition 9.6 (Snap-back repellers). *For a noninvertible map f with repelling fixed point p, a snap-back repeller is a homoclinic orbit $\{z_k\}_{k \in I}$ for which every point in the orbit has $Df(z_k)$ an isomorphism.*

Namely, snap-back repellers are homoclinic orbits of repelling fixed points, which are contained in the zero-dimensional stable manifold.

Example 9.7 (Iterated function systems). All of the examples listed above are noninvertible maps. An iterated function system [4] is a simple example of a multivalued map.

Definition 9.8 (Iterated function systems). *An iterated function system is a complete metric space with a finite set of smooth contractions $\{w_n, n = 1, \ldots, N\}$.*

Of interest is the dynamics of the multivalued map $\cup_{n=1}^{N} w_n$. Under the assumption that for all x and $i \neq j$, $\omega_i(x) \neq \omega_j(x)$, an iterated function system forms a smooth relation. The whole space has a trivial splitting with no unstable directions. Thus the entire space has hyperbolic structure.

The stable manifold theorem says that every point z_o with an infinite forward orbit $\{z_k\}_{k\geq 0}$ has a neighborhood in which every point w has a forward orbit converging to the orbit $\{z_k\}_{k\geq 0}$. For iterated function systems on a compact space, this stable manifold turns out to be the whole space, the orbit depending only on the sequence of contractions chosen. The theorem also says that the unstable manifold for a backward orbit is zero-dimensional.

Using the above, we can recover information about which sequence of contractions in F converges to a given point, as in Barnsley's Chaos Game [4]. Let $(x, y) \in F$. Then $y = \omega_j(x)$ for some j. Notice that for sufficiently small δ, all points in $F \cap B_\delta(x, y)$ are also of the form $(z, \omega_j(z))$ for the same j. Thus for a sufficiently small pseudo-orbit, we can recover the bi-infinite sequence of contractions from F which give the shadowing orbit. Also, to every backward sequence of contractions, there corresponds a unique point with the sequence corresponding to its backward orbit. However, there may be many forward and backward orbits through such a point.

In [4], there is a proof of a version of the shadowing lemma for iterated function systems which are the union of invertible contractions. Here, we do not assume that contractions are invertible, but the main difference is in the framework and approach to looking at iterated function systems.

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