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DYNAMICS OF NONINVERTIBILITY IN DELAY EQUATIONS

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Abstract. Models with a time delay often occur, since there is a naturally occurring delay in the transmission of information. A model with a delay can be noninvertible, which in turn leads to qualitative differences between the dynamical properties of a delay equation and the familiar case of an ordinary differential equation. We give specific conditions for the existence of noninvertible solutions in delay equations, and describe the consequences of noninvertibility.

1. **Introduction.** Delay differential equations arise in many important applications. For example, they are inherent in biology due to finite information transmission times. Of interest are delay equations of the form

$$\frac{dx}{dt}(t) = F(x(t-\tau)) - \gamma x(t), \tag{1}$$

where $\gamma > 0$, and initial conditions are specified by $x(t) = \phi(t)$ for $t \in [t_0 - \tau, t_0]$. For example, this is the form of the celebrated Mackey-Glass equation for dynamical diseases [27, 28, 32], where $x(t)$ describes the white blood cell density at time t, γ is the blood cell death rate, and F is a nonlinear function for production rate as a function of density. The delay results from stem cell maturation time. The following F corresponds to Equation 4b in Mackey and Glass's original paper [27]:

$$F(x) = ax\frac{\theta^k}{\theta^k + x^k}, \tag{2}$$

where $a > 0$, $\theta > 0$, and $k > 1$ are shape parameters (See Figure 1, top left).

Delay equations of the same form are also used as models for dynamical diseases in references [18, 19, 20, 29, 35]. Similar delay equations are used to describe irregular breathing in adults [13, 27, 36, 40], and in models for lasers [3, 21].

It is well known that for an ordinary differential equation $\frac{dx}{dt} = F(x)$, where F is Lipschitz, two different initial states display distinct solutions both forwards and backwards in time. Thus looking at the initial state of a quantity modeled by an ODE, it is possible to fully and uniquely describe both the future and the history of its time evolution. This non-overlapping nature of solutions is an important property, as it affects not only the ability to know the past, but also the qualitative behavior of dynamical structures of the model, such as global invariant manifolds, synchronization manifolds, and basins of attraction.

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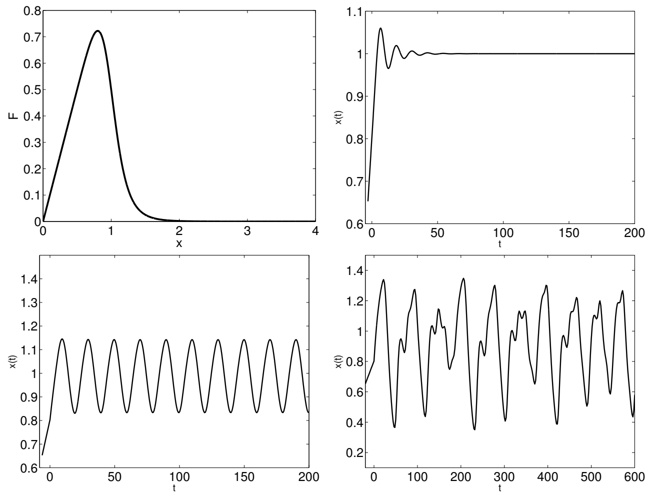

FIGURE 1. Top left: Graph of the function F in Equation (2), where $a = 1$, $\theta = 1$, and $k = 10$. Top right and bottom: Solutions to the Mackey-Glass equation (Equations (1) and (2)) for three different delays: $\tau = 3$ (top right), $\tau = 6$ (bottom left), and $\tau = 20$ (bottom right). In each case, $a = 0.2, \theta = 1, k = 10.0$, and $\gamma = 0.1$, and the initial condition is the linear function $x(t) = 0.015t+0.8027$ on $[-\tau, 0]$.

Unlike ordinary differential equations, the initial value problem in Equation (1) can have solutions which are not one-to-one no matter how smooth F is assumed to be (*cf.* Ch. 2 and 3 of [14]). We call this nonuniqueness *noninvertibility of solutions*. In this paper, we classify the noninvertibility of solutions of equations of the following form:

$$\frac{dx}{dt}(t) = H(x(t), x(t-\tau)), \tag{3}$$

where $x(t) = \phi(t)$ on $[t_0 - \tau, t_0]$, and $H : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. We then apply these results to the specific case given in Equation (1).

Many of the major dynamical results from ODEs and invertible iterated maps carry over to noninvertible differential equations and noninvertible maps, including the shadowing lemma, the existence of local invariant manifolds, and Poincaré sections [17, 16, 15, 23, 24, 38, 39]. However, the presence of noninvertibility can give rise to fundamentally different dynamical properties than those seen in the ODE case. Examples occur in adaptive control systems [1, 8, 9], neural networks [33], numerical methods [11, 26], and synchronization [2, 4, 5, 6, 22, 37]. Further theoretical treatments of noninvertible dynamics in finite dimensions can be found in citations [10, 34] and references therein.

Lorenz's work and the followup paper of Frouzakis, Kevrekidis, and Peckham [11, 26] discuss the connection between the onset of noninvertibility and chaotic behavior within an attractor in a specific planar map. When the attractor intersects the singular set of the map, there is an increase in the attractor's complexity. Since the singular set of a finite dimensional map is typically codimension one, this type of bifurcation is expected in an open set of one-parameter families.

Knowing when there is noninvertibility of solutions is crucial to classifying the dynamics of solutions. In fact, for the initial value problem (3), the existence of noninvertible solutions is generically a necessary condition for the existence of chaotic solutions: Mallet-Paret and Sell have shown that if the nonlinearity F is monotone for the initial value problem given in Equation (3), then the Poincaré-Bendixson theorem holds, meaning that solutions are very simple and non-chaotic [31, 30]. That is, this type of delay equation, when monotone, behaves like a planar ordinary differential equation. Monotonicity for Equation (3) is defined to mean $\frac{\partial H(\xi,\eta)}{\partial \eta} \neq 0$ for all $(\xi, \eta) \in \mathbb{R}^2$. In the current paper, we show that monotonicity implies invertibility. That is, a backwards continuation of solutions is unique (though it may not exist). In fact, monotonicity and invertibility are generically equivalent: If the derivative above is equal to zero, along with some generic conditions on the second partial derivatives which are given in Theorem 2, then there exist noninvertible solutions.

In contrast to autonomous ordinary differential equations, for which three dimensions are required for chaotic solutions, solutions to delay equations with one spatial dimension can exhibit chaotic behavior. For example, the Mackey-Glass equation above has chaotic solutions. In addition, Gedeon and Lani-Wayda have each given examples to show that chaotic solutions occur even in simple cases of non-monotone delay equations [12, 25].

The paper proceeds as follows: In Sections 2 and 3, we begin with some examples of delay equations with noninvertible solutions. In Section 4, we give necessary conditions for a solution to be noninvertible for the initial value problem in Equation (3). These are specific geometric properties necessary for a solution to have two distinct backward continuations. In Section 5, we give stronger results for the initial value problem in Equation (1). Namely, we show that if a solution has two distinct backward continuations starting at time t_m, then both continuations have critical points at times $t_m - k\tau$, where the degree of the critical points increases with k. Section 6 contains concluding remarks.

2. **Mackey-Glass example.** The Mackey-Glass equation given by Equations (1) and (2) with $t_0 = 0$ is a delay equation displaying noninvertibility. For small values of the delay parameter τ, solutions to the Mackey-Glass equation converge to a stable equilibrium. (See Figure 1a.) When τ is larger, solutions converge to a periodic oscillatory solution with period between 2τ and 4τ [13]. (See Figure 1b.) When τ grows still larger, solutions are known to become chaotic. (See Figure 1c.) Note that this chaotic behavior indicates noninvertibility: By the results of Mallet-Paret and Sell [31, 30], chaotic behavior does not occur for delay equations of this form satisfying monotonicity (invertibility). Since the function F is unimodal, the results of the next two sections show directly that for the Mackey-Glass equation, there are noninvertible solutions. Figure 2a shows two different initial states which result in the same solution from time zero onwards.

Note that for this example, we have chosen Mackey and Glass's nonlinearity from Equation 4b of their orginal paper [27]. Their Equation 4a is monotone increasing, which implies unique backwards continuations.

3. **Another example.** Assume that solutions $x(t)$ and $y(t)$ defined on $[t_0-\tau, t_0+\epsilon]$ have the property that there exists a $T > t_0 - \tau$ such that $x(t) = y(t)$ for all $t > T$. We denote by t_m the minimal value of T for which this statement holds.

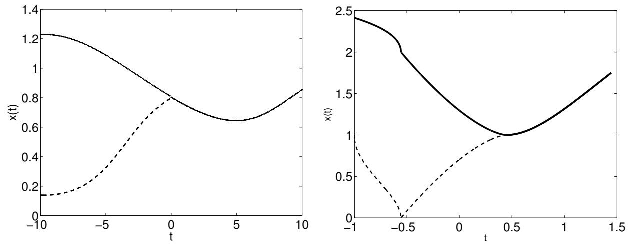

FIGURE 2. Left: Two solutions for the Mackey-Glass equation chosen so that $x(t) = y(t)$ for all $t \geq t_m = 0$. Parameter $\tau = 10$. All other parameters are the same as in the previous figure. Right: A pair of noninvertible solutions with $t_m > 0$ from Section 3.

In this paper, we concentrate on the case where $t_m > t_0$. At the end of Section 4, we show that the case of $t_0 - \tau < t_m \leq t_0$ is straightforward. We refer to $x(t)$ and $y(t)$ as *distinct* if there exists a t such that $x(t) \neq y(t)$. In the previous example, $x(t)$ and $y(t)$ were distinct solutions, and $t_m = t_0 = 0$. In the following example, $t_m > t_0 = 0$. See Figure 2b. Subsequent sections demonstrate that the distinction between $t_m = t_0$ and $t_m > t_0$ is extremely significant. This is due to the smoothness properties of solutions. Namely, functions as in Figure 2a may not be differentiable at $t = t_m = t_0$, whereas the functions in Figure 2b are all differentiable at $t = t_m > t_0$. The delay equation is the initial value problem in Equation (1). For simplicity $\gamma = 0$, $\tau = 1$, and $F(x) = x(2-x)$. A detailed calculation using forward integration shows that the following two continuous initial conditions lead to distinct solutions which are identical starting at time $t_m = 4/9$.

$$x(t) = \begin{cases} 1 + \frac{1}{2}\sqrt{4 - \frac{2(\frac{19}{3}+6t-3(t+\frac{14}{9})^2)}{\sqrt{-\frac{11}{3}-3t+3(t+\frac{14}{9})^2-(t+\frac{14}{9})^3}}} & \text{when } -1 \leq t \leq -5/9 \\ 1 + \sqrt{-\frac{2}{3} - 3t + 3(t+\frac{5}{9})^2 - (t+\frac{5}{9})^3} & \text{when } -5/9 \leq t \leq 0. \end{cases}$$

$$y(t) = \begin{cases} 1 - \frac{1}{2}\sqrt{4 + \frac{2(\frac{19}{3}+6t-3(t+\frac{14}{9})^2)}{\sqrt{-\frac{11}{3}-3t+3(t+\frac{14}{9})^2-(t+\frac{14}{9})^3}}} & \text{when } -1 \leq t \leq -5/9 \\ 1 - \sqrt{-\frac{2}{3} - 3t + 3(t+\frac{5}{9})^2 - (t+\frac{5}{9})^3} & \text{when } -5/9 \leq t \leq 0. \end{cases}$$

4. **Noninvertible solutions.** In this section, we give a specific set of conditions for the existence of noninvertible solutions for delay equations. For the initial value problem in Equation (3), assume H and ϕ are continuous. Then there exists $\epsilon > 0$ and a unique continuous function $x(t)$ such that $x(t)$ has the prescribed initial value $\phi(t)$ on $[t_0 - \tau, t_0]$ and is a solution to the delay equation (3) on $[t_0, t_0 + \epsilon]$. See [7] or [14].

Henceforth, without loss of generality, we simplify the notation and assume that $t_0 = 0$. Let $H(r,s)$ be a smooth C^k function with $k \geq 2$. Assume $\frac{\partial H}{\partial s}(r_0, s_0) = 0$, and $\frac{\partial^2 H}{\partial s^2}(r_0, s_0) \neq 0$. Then by the implicit function theorem for any (r, s_1) near (r_0, s_0) with $\frac{\partial H}{\partial s}(r, s_1) \neq 0$, there exists $s_2 \neq s_1$ such that $H(r, s_2) = H(r, s_1)$. The following two theorems show that all noninvertible solutions to Equation (3) arise from this and other similar equalities of H values. The first theorem gives necessary conditions to guarantee noninvertibility of solutions.

Theorem 1. *Suppose that $x(t)$ and $y(t)$ are distinct solutions to Equation (3) with continuous initial conditions, $H(r,s)$ is C^1, and that there is some least time $t_m \geq 0$ for which $x(t) = y(t)$ for all $t \geq t_m$. Then $H(x(t), x(t-\tau)) = H(y(t), y(t-\tau))$ for $t \geq t_m$, and $\frac{\partial H}{\partial s}(x(t_m+\tau), x(t_m)) = \frac{\partial H}{\partial s}(y(t_m+\tau), y(t_m)) = 0$.*

Proof. The proof of this theorem uses a similar technique as was used to show backwards uniqueness in Mallet-Paret and Sell [30].

Under the hypotheses of the theorem, the functions $x(t)$ and $y(t)$ are differentiable for $t > 0$, since they solve a differential equation with continuous initial conditions. This implies that $x'(t) = y'(t)$ for $t > t_m$. Since $x(t)$ and $y(t)$ solve Equation (3), this is equivalent to the statement that $H(x(t), x(t-\tau)) = H(y(t), y(t-\tau))$ for $t > t_m$. By the continuity of H, this equation holds for $t = t_m$ as well. Since t_m is the smallest time such that $x(t)$ and $y(t)$ are equal for all greater t, $x(t)$ and $y(t)$ are not identical for any time interval with right end point t_m. If $\frac{\partial H}{\partial s}(x(t_m+\tau), x(t_m)) \neq 0$, then by the implicit function theorem, for $t = t_m + \tau$ the differential equation (3) is locally uniquely invertible, which implies that $x(t) = y(t)$ for a neighborhood of t_m, which contradicts the assumption on the minimality of t_m. $\square$

Thus although there may be infinitely many noninvertible solutions, any pair of noninvertible solutions meet at the point at which the function H has a critical point with respect to the second variable. In the following theorem, we obtain stronger results for the case $t_m > 0$.

Theorem 2. *Assume that $x(t)$ and $y(t)$ are distinct solutions to Equation (3) with continuous initial conditions, H is C^{L+2}, $L \geq 0$, and that there exists a least time $t_m > 0$ such that $x(t) = y(t)$ for $t \geq t_m$. Let $S = \left\{(r,s) : \frac{\partial H}{\partial s} = 0\right\}$. Let $(r_0, s_0) = (x(t_m+\tau), x(t_m))$. Assume that $\frac{\partial^2 H}{\partial s^2}(r_0, s_0) \neq 0$. Thus we can write S as the graph of a C^{L+1} function g. That is, $S = \{(r,s) : s = g(r)\}$. Then $x'(t_m) = y'(t_m) = g'(r_0)x'(t_m+\tau)$. In fact, as long as $x(t)$ and $y(t)$ are C^{L+1} at t_m, for all $1 \leq j \leq L$, $\frac{d^j x}{dt^j}(t_m) = \frac{d^j y}{dt^j}(t_m) = \frac{d^j g(x(t_m+\tau))}{dt^j}$.*

Proof. Functions $x(t)$ and $y(t)$ are equal for all $t \geq t_m$ but are not identical for $t < t_m$. Thus there exists a sequence of times t_k converging to t_m such that $x(t_k) \neq y(t_k)$, but for k sufficiently large, $H(x(t_k+\tau), x(t_k)) = H(x(t_k+\tau), y(t_k))$ (since $y(t_k+\tau) = x(t_k+\tau)$). Define the curve $\Gamma = \{(t, g(x(t+\tau)))\}$. Combining the equality of the H values, $\frac{\partial^2 H}{\partial s^2}(r_0, s_0) \neq 0$, and the fact that $x(t_m) = y(t_m)$ for t_k sufficiently close to t_m, $x(t_k)$ and $y(t_k)$ are the two local inverses of $H(x(t_k + \tau), x(t_k))$, one lying above and one lying below the curve Γ for $t = t_k$. Thus for fixed but sufficiently large k, the two points $x(t_k)$ and $y(t_k)$ are on opposite sides of the curve Γ. See Figure 3. There are two possibilities:

1) Assume that there is a time t_L near t_m, with $t_L < t_m$ such that for each fixed t such that $t_L < t < t_m$, the point $g(x(t+\tau))$ lies between $x(t)$ and $y(t)$. Since the two functions $x(t)$ and $y(t)$ both have the same derivative at t_m, they must be tangent to the curve $(t, g(x(t+\tau)))$ at t_m. That is, $x'(t_m) = y'(t_m) = g'(x(t_m+\tau))x'(t_m+\tau)$.

Working iteratively, a Taylor series expansion of $x(t)$ and $y(t)$ at t_m gives the higher derivative cases: Let $\alpha(t) = x(t) - g(x(t+\tau))$, and $\beta(t) = y(t) - g(x(t+\tau))$. The functions α and β are C^{L+1} functions. We showed above that for $t < t_m$ but near t_m, α and β have opposite signs. We also know that $\alpha'(t_m) = \beta'(t_m) = 0$.

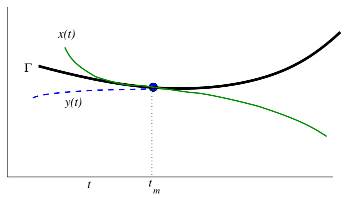

FIGURE 3. A depiction of solutions $x(t)$ and $y(t)$ and singularity curve Γ in the proof of Theorem 2.

Assume that $\alpha^{(n)}(t_m) = \beta^{(n)}(t_m) = 0$ for all $1 \leq n \leq N-1$, where $N - 1 < L$. A Taylor series expansion gives that for some $c \in (t, t_m)$, $\alpha(t) = \alpha^{(N)}(t_m)\frac{(t-t_m)^N}{N!} + \alpha^{(N+1)}(c)\frac{(t-t_m)^{(N+1)}}{(N+1)!}$. The same formula holds for β in place of α. By the smoothness properties of α and β, for t sufficiently close to t_m, the sign of $\alpha(t)$ is determined by the sign of $\alpha^{(N)}(t_m)$. Functions α and β are of opposite signs, but $\alpha^N(t_m) = \beta^N(t_m)$, so $\alpha^{(N)}(t_m) = 0$.

2) If both functions oscillate across the curve S arbitrarily close to t_m, then by the mean value theorem, there is a sequence of points t_j converging to t_m such that $x'(t_j)$ converges to the mean value of the curve Γ. Therefore $x'(t_m)$ is equal to the tangent to S. The proof of the equality of higher derivatives is similar. □

We end this section with a comment on sufficient conditions for noninvertible solutions. If a given solution satisfies all of the above necessary conditions for noninvertibility at a time $t_m > 0$, it is still not easy to guarantee the existence of two distinct backward continuations. However, it is very straightforward to write down sufficent conditions for $-\tau < t_m \leq 0$. Namely, if a solution $x_1(t)$ with continuous initial condition $\phi_1(t)$ is such that $\frac{\partial H}{\partial s}(x(t_m + \tau), x(t_m)) = 0$ and $\frac{\partial^2 H}{\partial s^2}(x(t_m+\tau), x(t_m)) \neq 0$, then take $\phi_2(t)$ to be the other local preimage of $H(x(t+\tau), x(t))$. If $\phi_2(t)$ exists for all $-\tau < t < 0$, then there is a second solution $x_2(t)$ which is identical to $x_1(t)$ for all $t \geq t_m$.

5. **Noninvertibility and one-dimensional maps.** In this section, we apply the general results from the previous section to the broad class of delay equations given in Equation (1), where F is any smooth nonlinear function. In this case, we are able to extend the results from the previous section to show that the values of noninvertible solutions are given by a one-dimensional iterated map. The following are restatements of Theorems 1 and 2.

Corollary 1. *Suppose that $x(t)$ and $y(t)$ are distinct solutions to Equation (1) with continuous initial conditions and F smoooth, and that for some least $t_m \geq 0$, $x(t) = y(t)$ for all $t \geq t_m$. Then $F(x(t)) = F(y(t))$ for $t \geq t_m - \tau$, and $F'(x(t_m)) = F'(y(t_m)) = 0$.*

Corollary 2. *Assume that $x(t)$ and $y(t)$ are distinct solutions to Equation (1) with continuous initial conditions and F smooth, and that there exists a least time $t_m > 0$ such that $x(t) = y(t)$ for $t \geq t_m$. Then $x'(t_m) = y'(t_m) = 0$. In fact, if x and y are $(k+1)$-times differentiable, then the first k derivatives vanish: $x^{(j)}(t_m) = y^{(j)}(t_m) = 0$ for $1 \leq j \leq k$.*

This result makes it straightforward to recognize possible noninvertible solutions by looking at the graph of $x(t)$ near t_m. Namely, one looks for critical points of the solution at a height equal to a critical point of F. The problem with this result is that since we do not know when t_m occurs, we must look at the graphs of solutions for arbitrarily large times. To remedy this, the next result shows that noninvertible solutions have critical points backwards from t_m with period τ.

Theorem 3. *Assume $x(t)$ and $y(t)$ are noninvertible solutions to Equation (1) with continuous initial conditions such that $t_m > 0$ is the least time for which $x(t) = y(t)$ for all $t \geq t_m$. Assume that F is a generic smooth function. For a fixed positive integer n, if $t_m > n\tau \geq 0$, then $x(t)$ and $y(t)$ have critical points at $t_m, t_m - \tau, t_m - 2\tau, \ldots, t_m - n\tau$. These critical points are of degrees $n+1, n, \ldots, 1$ respectively.*

Proof. In order to prove Theorem 3, we use the following standard result on the smoothness of solutions of a delay equation. See [14]: If the initial function $\phi(t)$ is continuous, then the solution to the delay equation in Equation (3) is C^1 for $t > 0$. It is C^2 for $t > \tau$. In general, the solution is C^k for $t > (k-1)\tau$.

Assume that n is the largest integer such that $0 \leq n\tau < t_m$. From the first paragraph of this proof, we know that $x^{(j)}(t_m)$ exists for all j such that $1 \leq j \leq n+1$. From Corollary (2), the derivatives up to n are all zero. Define

$$G(x) = F(x)/\gamma.$$

The following is our generic assumption on F: The critical points of G are nondegenerate and not periodic under G. It is straightforward to show that for a differentiable function $G : \mathbb{R} \to \mathbb{R}$, generically the critical points of G are nondegenerate, not periodic, and no two critical points are in the same orbit.

Let $k = 0$. By the previous paragraph, x has a degree $(n+1-k)$ critical point at $t_m - k\tau$. Further, by Equation (1), $F(x(t_m - \tau)) - \gamma x(t_m) = x'(t_m) = 0$, which implies that $G(x(t_m - \tau)) = x(t_m)$.

Let k be such that $1 \leq k \leq n$. Make the inductive hypothesis that for all $j = 0, \ldots, k$, x has a degree $(n+1-j)$ critical point at $t_m - j\tau$, and that for $j \geq 1$,

$$G(x(t_m - (j+1)\tau)) = x(t_m - j\tau).$$

Therefore the $(k+1)^{th}$ iterate of $x(t_m - (k+1)\tau)$ is a critical point. By the generic hypothesis, $x(t_m - (k+1)\tau)$ is not a critical point for G, which implies that $F'(x_m - (k+1)\tau) \neq 0$. Differentiating Equation (1) once gives

$$x''(t_m - k\tau) = x'(t_m - (k+1)\tau)F'(x(t_m - (k+1)\tau)) - \gamma x'(t_m - k\tau).$$

By the induction hypothesis, $x''(t_m - k\tau) = x'(t_m - k\tau) = 0$. Therefore

$$x'(t_m - (k+1)\tau) = 0.$$

Combining $x'(t_m - (k+1)\tau) = 0$ with Equation (1) at $t = t_m - (k+1)\tau$ gives

$$G(x(t_m - (k+2)\tau)) = x(t_m - (k+1)\tau).$$

To show that there is a degree $(n+1-(k+1))$ critical point at $t_m - (k+1)\tau$, differentiate Equation (1) j times where j is successively taken to be $2, \ldots, n+1-k$, and evaluate at $t = t_m - k\tau$. In each case, since it is already known that for $i < j-1$, $x^{(i)}(t_m - k\tau) = 0$, the equation reduces to

$$0 = x^{(j-1)}(t_m - (k+1)\tau)F'(x(t_m - (k+1)\tau)).$$

The statement $F'(x(t_m - (k+1)\tau)) \neq 0$ implies that $x^{(j-1)}(t_m - (k+1)\tau) = 0$.

This completes the proof. $\square$

The following corollary follows from the proof of the previous theorem. It states that the height of the critical points is determined by iteration under a one-dimensional map.

Corollary 3. *Let $G(x) = F(x)/\gamma$, where F is a smooth function with nondegenerate critical points, and γ is the constant from Equation (1). Assume $x(t)$ and $y(t)$ are noninvertible solutions with continuous initial conditions such that t_m is the first time for which $x(t) = y(t)$ for $t \geq t_m > 0$. The sequence of critical points described in Theorem 3 is a sequence of preimages of $x(t_m)$ under G: That is, for $k = 0, 1, \ldots, n$, $G(x(t_m - (k+1)\tau)) = x(t_m - k\tau)$.*

Using this result, the presence of noninvertibility can be translated into a condition on the parameter γ.

Corollary 4. *Under the hypotheses of Theorem 3, if there are noninvertible solutions to the initial value problem (1) for $t_m > 0$, then there exists a critical point x_m of F such that x_m has two preimages under F/γ.*

For example, in the case of the nonlinearity used for the original Mackey-Glass equation given in Equation (2), this corollary gives the condition that there are no noninvertible solutions for $\gamma > a(k-1)/k$ with $t_m > 0$.

6. **Conclusion.** Systems which depend on time delays are prevalent in biology. For example, all real neural systems have propagation delays, and population fluctuations often are cyclical with delayed feedback. In contrast to systems modeled by ordinary differential equations, solutions for a delay equation may overlap, meaning that two distinct solutions may become equal starting at a time $t_m \geq 0$. That is, looking at the current state, the past may be unknowable. This noninvertibility has a fundamental effect on the dynamical aspects of the system.

This paper has given a classification of the behavior of noninvertible solutions for the initial value problem in Equation (3). Specifically, the k^{th} derivative of H evaluated at the noninvertible point $t = t_m$ minus $k\tau$ is specified by the formula given in Theorem 2. For the initial value problem in Equation (1), where F is an arbitrary smooth function, this formula simplifies to saying that every $k\tau$ units backwards from $t = t_m$, the solution has a critical point with increasingly high degree.

Such conditions for the onset of noninvertibility in a finite dimensional attractor are important since they lead to a change in the structure of the attractor. For example, Lorenz showed a two-dimensional case in which noninvertibility resulted in an increase in the attractor's complexity [26].

The class of noninvertible solutions for delay equations is extremely specific and typically does not occur in direct simulations. We conjecture that unlike the case of finite dimensional noninvertible maps, a bifurcation to increased complexity of an attractor is not observable (i.e. non-generic) for families of delay equations. We further conjecture that for a fixed generic family of delay differential equations, there exists a global bound on the least time t_m described above. There is evidence to indicate that just as in the finite dimensional case, noninvertibility and chaotic behavior are linked for solutions to delay equations. The lack of chaos in the monotone (invertible) case for a higher dimensional generalization of the initial value problem in Equation (1) was shown by Mallet-Paret and Sell, whereas Gedeon and Lani-Wayda found chaos in non-monotone examples.

Intrinsic dynamical structures for differential equations possessing noninvertibility significantly differ from the invertible case. Such effects are independent of noise, and cannot be reduced even by careful experimentation or noise reduction techniques. For example, the presence of noninvertibility fundamentally changes the invariant manifolds such that they are no longer manifolds, may self-intersect, and may not even have a consistent dimension. In the specific case of coupled systems, synchronization sets for ordinary differential equations are themselves a type of invariant manifold. We have already shown that in the finite dimensional noninvertible case, synchronization sets become multivalued. This hampers the effectiveness of many of the standard synchronization detection methods [37]. In the delay equations case as well, they will no longer retain most of the nice properties seen for the case of coupled ordinary differential equations.

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