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Generalized Lorenz equations on a three-sphere

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Abstract. Edward Lorenz is best known for one specific three-dimensional differential equation, but he actually created a variety of related N-dimensional models. In this paper, we discuss a unifying principle for these models and put them into an overall mathematical framework. Because this family of models is so large, we are forced to choose. We sample the variety of dynamics seen in these models, by concentrating on a four-dimensional version of the Lorenz models for which there are three parameters and the norm of the solution vector is preserved. We can therefore restrict our focus to trajectories on the unit sphere S^3 in $\mathbb{R}^4$. Furthermore, we create a type of Poincaré return map. We choose the Poincaré surface to be the set where one of the variables is 0, i.e., the Poincaré surface is a two-sphere S^2 in $\mathbb{R}^3$. Examining different choices of our three parameters, we illustrate the wide variety of dynamical behaviors, including chaotic attractors, period doubling cascades, Standard-Map-like structures, and quasiperiodic trajectories. Note that neither Standard-Map-like structure nor quasiperiodicity has previously been reported for Lorenz models.

1 Three Lorenz systems

Edward Lorenz introduced polynomial systems of differential equations in a series of papers [1–7]. These equations all are dissipative, in the sense that all trajectories eventually end up in a finite size ball. The dynamics are chaotic for some parameter choices. While Lorenz justified these systems in terms of their being highly simplified meteorological or fluid-flow models, they are so severely simplified as to have little concrete practical use in meteorology. Regardless, in the intervening time, Lorenz equations have had a significant scientific impact, and their importance resides largely in their interesting dynamical behaviors. In this paper, we develop an overlying framework for the Lorenz systems which additionally produces a whole family of related differential equations. We call systems that fit into this framework generalized Lorenz

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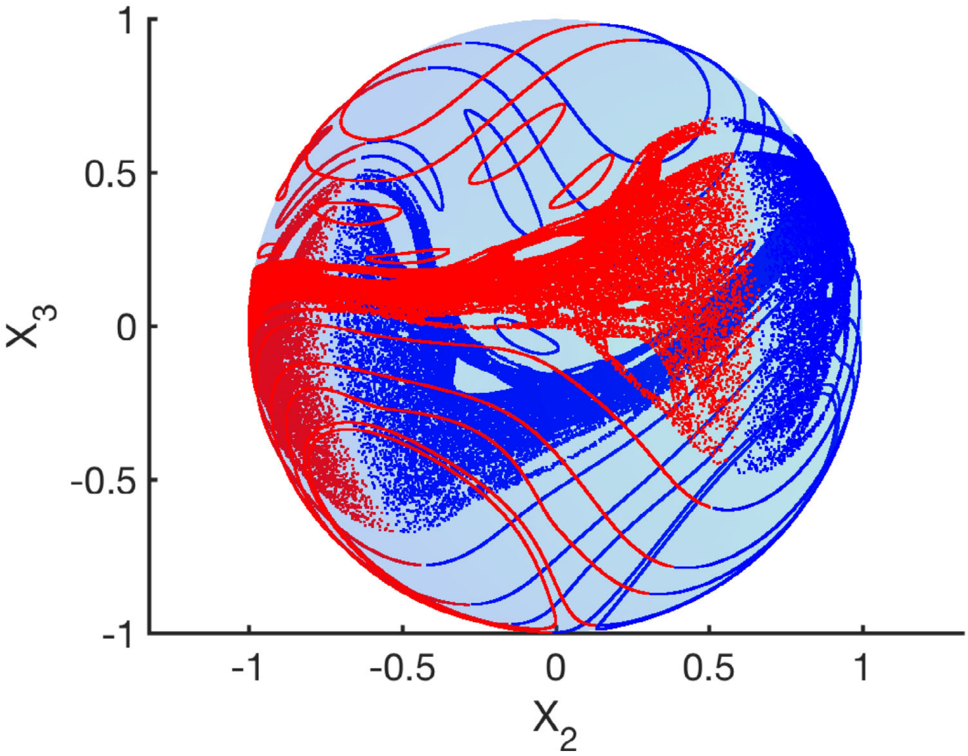

Fig. 1. Quasiperiodicity and chaos in the Poincaré Map on S^2 for GLE-4 with section $X_1 = 0$. Here we investigate trajectories of the 4-dimensional Generalized Lorenz Equation GLE-4, (given in Eq. (1) shown in this image for $\beta = 6$, $\rho = 8$, and $\gamma = 0$). We use initial conditions for which $\|\mathbf{X}\| = 1$. Since the norm $\|\mathbf{X}(t)\|$ remains constant, trajectories remain on the unit sphere, denoted S^3. To reduce the dimension by one, we only plot trajectories at those times when $X_1(t) = 0$, giving us a type of Poincaré map on the two-dimensional sphere S^2 shown here, given by $X_2^2 + X_3^2 + X_4^2 = 1$. We plot a Poincaré map point (X_2, X_3) in a lighter color (red online) when the derivative X_1' is positive and darker color (blue online) when negative. The colors are brighter (red and blue online) on the upper hemisphere $X_4 > 0$ and dull when $X_4 < 0$. Ten trajectories are shown here, nine of which are quasiperiodic. The full trajectory of each of the nine is a torus and here we show only the smooth curves where the torus intersects $X_1 = 0$. There is also one chaotic trajectory which can be seen wrapping around S^2 where $X_3 \approx 0$.

equations or Lorenz-like models. We start by developing the family of generalized Lorenz equations.

A generalized Poincaré map. The trajectories can be viewed using a Poincaré return map; see Figure 1. Using a Poincaré map reduces the dimension by 1, reducing the 3-sphere to a 2-sphere when we use the Poincaré surface such as $X_1 = 0$. Of course it is possible that even if a trajectory has its initial point on this surface, it may never return to the surface, such as might occur if there is an attractor that does not intersect the surface. In fact, an example of this phenomenon occurs for Poincaré's original return map, which was created for the circular restricted three-body problem. That surface of section only captures trajectories that cross the line between the two major bodies. The Lagrange Points L4 and L5 are equilibria that form equilateral