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UNEXPECTEDLY LINEAR BEHAVIOR FOR THE CAHN–HILLIARD EQUATION*

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Abstract. This paper gives theoretical results on spinodal decomposition for the Cahn–Hillard equation. We prove a mechanism which explains why most solutions for the Cahn–Hilliard equation starting near a homogeneous equilibrium within the spinodal interval exhibit phase separation with a characteristic wavelength when exiting a ball of radius R. Namely, most solutions are driven into a region of phase space in which linear behavior dominates for much longer than expected.

The Cahn–Hilliard equation depends on a small parameter ε, modeling the (atomic scale) interaction length; we quantify the behavior of solutions as $\varepsilon \to 0$. Specifically, we show that most solutions starting close to the homogeneous equilibrium remain close to the corresponding solution of the linearized equation with relative distance $O(\varepsilon^{2-n/2})$ up to a ball of radius R in the $H^2(\Omega)$-norm, where R is proportional to $\varepsilon^{-1+\varrho+n/4}$ as $\varepsilon \to 0$. Here, $n \in \{1, 2, 3\}$ denotes the dimension of the considered domain, and $\varrho > 0$ can be chosen arbitrarily small. Not only does this approach significantly increase the radius of explanation for spinodal decomposition, but it also gives a clear picture of how the phenomenon occurs.

While these results hold for the standard cubic nonlinearity, we also show that considerably better results can be obtained for similar higher order nonlinearities. In particular, we obtain $R \sim \varepsilon^{-2+\varrho+n/2}$ for every $\varrho > 0$ by choosing a suitable nonlinearity.

Key words. Cahn–Hilliard equation, spinodal decomposition, phase separation, pattern formation, linear behavior

AMS subject classifications. 35K35, 35B05, 35P10

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1. Introduction.

A particularly intriguing phenomenon in the study of binary alloys is *spinodal decomposition* [8]; namely, if a homogeneous high-temperature mixture of two metallic components is rapidly quenched below a certain lower temperature, then a sudden phase separation sets in. The mixture quickly becomes inhomogeneous and forms a fine-grained structure, more or less alternating between the two alloy components. Figure 1.1 shows a typical example of such a pattern.

In order to describe this phase separation process (as well as other phenomena) Cahn [6] and Cahn and Hilliard [9] proposed the fourth-order parabolic partial differential equation

$$
\begin{aligned}
u_t &= -\Delta(\varepsilon^2 \Delta u + f(u)) \quad \text{in} \quad \Omega, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega.
\end{aligned}
\tag{1.1}
$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary, $n \in \{1, 2, 3\}$, and the function $-f$ is the derivative of a double-well potential F, the standard example being the cubic function $f(u) = u - u^3$. Furthermore, ε is a small positive parameter modeling interaction length. In this formulation, the variable u represents the concentration of one of the two components of the alloy, subject to an

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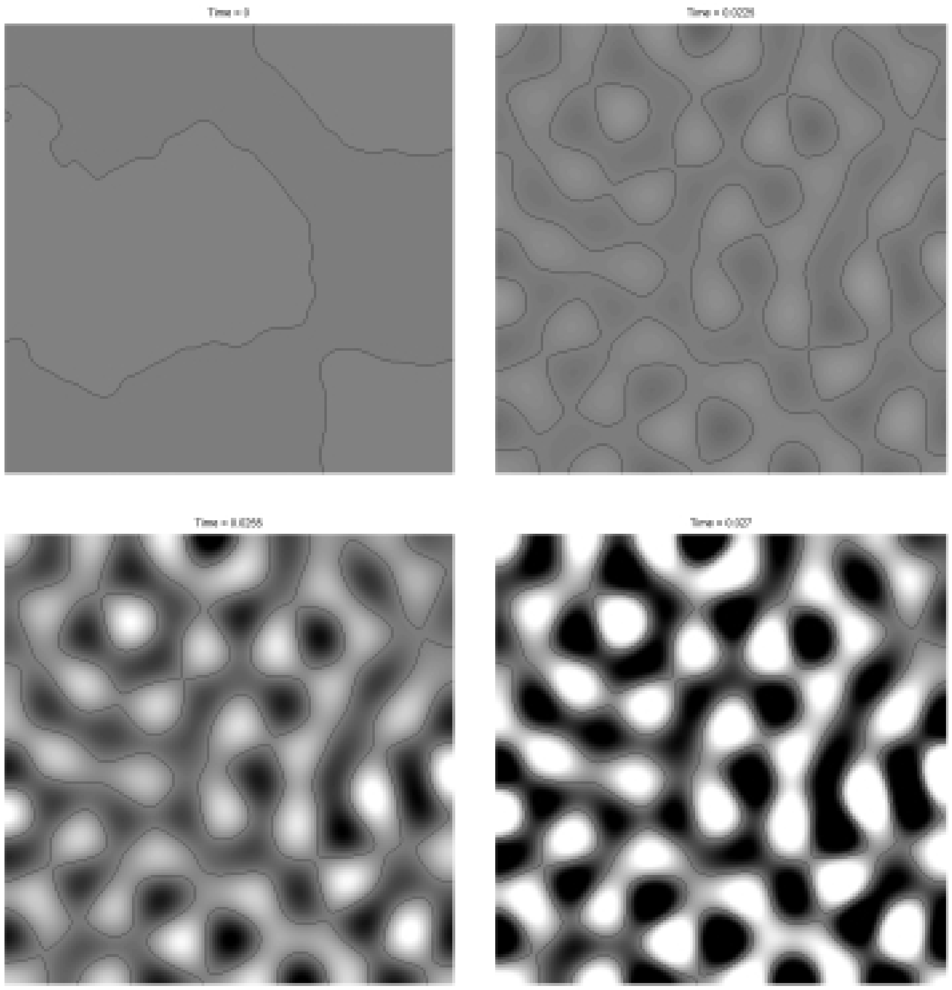

FIG. 1.1. *Spinodal decomposition in two dimensions for* $\varepsilon = 0.02$.

affine transformation such that the concentration 0 or 1 corresponds to u being -1 or 1, respectively. The Cahn–Hilliard equation is *mass-conserving*, i.e., the total concentration $\int_\Omega u(t,x)\,dx$ remains constant along any solution u. Moreover, (1.1) is an $H^{-1}(\Omega)$-*gradient system* with respect to the Van Der Waals free energy functional

$$E_\varepsilon[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} \cdot |\nabla u|^2 + F(u) \right) dx,$$

where F is the above-mentioned primitive of $-f$; see Fife [19].

Every constant function $\bar{u}_o \equiv \mu$ is a stationary solution of (1.1). Furthermore, this equilibrium is unstable if μ is contained in the *spinodal interval*. This is the (usually connected) set of all $\mu \in \mathbb{R}$ such that $f'(\mu) > 0$. Thus, if μ lies in the spinodal interval, any orbit originating near $\bar{u}_o$ is likely to be driven away from $\bar{u}_o$. In this paper, we prove the exact mechanism which explains precisely how this driving away process occurs. Basically, solutions starting near the equilibrium are driven into a region of phase space in which the linear terms dominate the behavior.

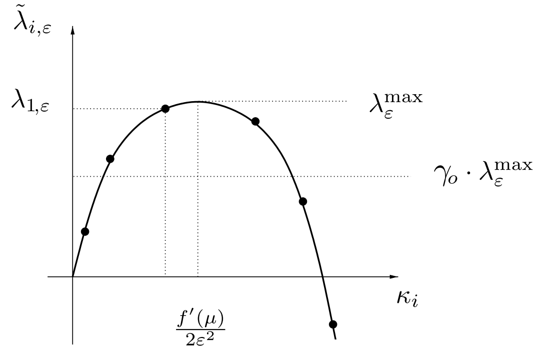

FIG. 1.2. *Eigenvalues of the linearization A_ε.*

There have been many works in the physics literature dealing with spinodal decomposition and how it is modeled by the Cahn–Hilliard equation. We refer the reader, for example, to Cahn [7, 8], Hilliard [22], Langer [25], Elder and Desai [15], Elder, Rogers, and Desai [16], and Hyde et al. [24]. There also exist numerous papers on numerical simulations of the Cahn–Hilliard equation; see, for example, Elliott and French [18], Elliott [17], Copetti and Elliott [11], Copetti [10], Bai et al. [2, 3], as well as our recent paper [31].

Mathematical treatments of spinodal decomposition in the Cahn–Hilliard equation have appeared in Grant [20] and Maier-Paape and Wanner [26, 28, 27]. Since spinodal decomposition is concerned with solutions of (1.1) originating near the homogeneous equilibrium $\bar{u}_o \equiv \mu$, it is not surprising that both of the above approaches crucially rely on the properties of the linearization of (1.1) at $\bar{u}_o$, given as follows:

$$\begin{aligned} v_t \;=\; A_\varepsilon v \;=\; -\Delta(\varepsilon^2 \Delta v + f'(\mu) v) \quad &\text{in} \quad \Omega \,, \\ \frac{\partial v}{\partial \nu} = \frac{\partial \Delta v}{\partial \nu} = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

Introducing

$$X = \left\{ v \in L^2(\Omega) : \int_\Omega v dx = 0 \right\}, \tag{1.3}$$

we can consider the operator $A_\varepsilon : X \to X$ with domain

$$D(A_\varepsilon) = \left\{ v \in X \cap H^4(\Omega) : \frac{\partial v}{\partial \nu}(x) = \frac{\partial \Delta v}{\partial \nu}(x) = 0 \,,\, x \in \partial\Omega \right\}. \tag{1.4}$$

It can be shown that this operator is self-adjoint. The spectrum of A_ε consists of real eigenvalues $\lambda_{1,\varepsilon} \ge \lambda_{2,\varepsilon} \ge \cdots \to -\infty$ with corresponding eigenfunctions $\varphi_{1,\varepsilon}, \varphi_{2,\varepsilon}, \ldots$. To further describe these eigenvalues, let $0 < \kappa_1 \le \kappa_2 \le \cdots \to +\infty$ and $\psi_1, \psi_2, \ldots$ denote the eigenvalues and eigenfunctions of the operator $-\Delta : X \to X$ subject to Neumann boundary conditions. Then the eigenvalues $\lambda_{i,\varepsilon}$ of A_ε are obtained by ordering the numbers

$$\tilde{\lambda}_{i,\varepsilon} = \kappa_i (f'(\mu) - \varepsilon^2 \kappa_i), \qquad i \in \mathbb{N}. \tag{1.5}$$

See Figure 1.2. The eigenfunctions $\varphi_{i,\varepsilon}$ are obtained from the eigenfunctions ψ_i through this ordering process in the obvious way and form a complete $L^2(\Omega)$-

orthonormal set in X. Moreover, the largest eigenvalue $\lambda_{1,\varepsilon}$ is of the order

$$\lambda_{1,\varepsilon} \sim \lambda_\varepsilon^{\max} = \frac{f'(\mu)^2}{4\varepsilon^2}, \qquad \text{and} \quad \lambda_{1,\varepsilon} \le \lambda_\varepsilon^{\max}. \tag{1.6}$$

See Maier-Paape and Wanner [26].

The strongest unstable directions are the ones corresponding to $\kappa_i \approx f'(\mu)/(2\varepsilon^2)$, and one would expect that most solutions of (1.2) originating near $\bar{u}_o \equiv \mu$ will be driven away in some unstable direction(s).

In order to deduce results about the dynamics of the nonlinear Cahn–Hilliard equation from the above linearization, Grant [20] and Maier-Paape and Wanner [26, 28] employed a dynamical approach. Equation (1.1) generates a nonlinear semiflow $T_\varepsilon(t)$, $t \ge 0$, on the affine space $\mu + X^{1/2}$, where $X^{1/2}$ denotes the Hilbert space

$$X^{1/2} = \left\{ v \in H^2(\Omega) \cap X : \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}. \tag{1.7}$$

The constant function $\bar{u}_o \equiv \mu$ is an equilibrium point for T_ε, and the linearization of T_ε at $\bar{u}_o$ is given by the analytic semigroup S_ε generated by A_ε.

For the above setting, Grant [20] described spinodal decomposition for one-dimensional domains Ω by showing the following. For generic small ε, most solutions of (1.1) starting in a specific neighborhood U_ε of $\bar{u}_o \equiv \mu$ stay close to the one-dimensional strongly unstable manifold of the equilibrium. This unstable manifold is tangent to the eigenfunction $\varphi_{1,\varepsilon}$ of the largest eigenvalue $\lambda_{1,\varepsilon}$. Furthermore, the two branches of the strongly unstable manifold converge to two equilibrium points of (1.1) which are periodic in space, and whose L^∞-norm is bounded away from 0 as $\varepsilon \to 0$. These equilibria can be interpreted as spinodally decomposed states. Thus over time, most solutions originating in U_ε grow near the spinodally decomposed states.

Grant's approach is not sufficient to explain spinodal decomposition in more than one dimension. His approach predicts evolution of orbits towards regular patterns which are not observed in practice. Maier-Paape and Wanner [26, 28] pointed out that this discrepancy is due to the fact that the size of the neighborhood U_ε in Grant's result is of the order $\exp(-c/\varepsilon)$. They proposed a different approach for explaining spinodal decomposition in all (physically relevant) dimensions. Their results consider solutions of (1.1) starting in a neighborhood U_ε with size proportional to $\varepsilon^{\dim\Omega}$. They prove that most solutions of (1.1) originating in U_ε exit a larger neighborhood $V_\varepsilon \supset U_\varepsilon$, also of the order $\varepsilon^{\dim\Omega}$, close to a dominating linear subspace, spanned by the eigenfunctions corresponding to a small percentage of the largest eigenvalues of A_ε. Its dimension is proportional to $\varepsilon^{-\dim\Omega}$.

The approach of Maier-Paape and Wanner is more successful than that of Grant in describing observed patterns. However, the result is not optimal. The size of the neighborhood V_ε is proportional to $\varepsilon^{\dim\Omega} \ll 1$ with respect to the $H^2(\Omega)$-norm, whereas the patterns they predict are observed even when the sup norm of the solution is of order 1. Moreover, according to Maier-Paape and Wanner [28, Remark 3.6], functions in the dominating subspace generally exhibit a sup norm of order 1 only if their $H^2(\Omega)$-norm is of the order ε^{-2}, which is increasing as $\varepsilon \to 0$. This difference in the orders is due to the fact that functions in the dominating subspace are oscillatory, i.e., exhibit large second derivative terms.

In this paper, we give an improved result to explain spinodal decomposition, using a new approach. For $f(u) = u - u^3$ and $\mu = 0$ our explanation applies to balls U_ε which are polynomial in ε, and V_ε of size proportional to $\varepsilon^{-1+\varrho+\dim\Omega/4}$, where $\varrho > 0$

is arbitrarily small. Note that this is a remarkable improvement over the previous two estimates, since the size of our starting domain is physically visible, and our explanation applies for an exit domain which is growing as $\varepsilon \to 0$. Furthermore, it gives more precise information about the behavior of solutions. Namely, we are able to show that spinodal decomposition is not merely a result of the dominance of a linear subspace; it is a result of the fact that most solutions are driven into a region of phase space in which the behavior is essentially linear. This is completely unexpected, as the nature of the equation throughout most of V_ε is highly nonlinear. Many solutions which stay in V_ε for some time show clearly nonlinear behavior during this time. It is only solutions starting near the equilibrium which are very likely to exhibit linear behavior while they remain in V_ε. This has been described in more detail in Sander and Wanner [31].

We are able to precisely quantify this linear regime in terms of the relative distance between the solutions to the linear and nonlinear equations. Neglecting technical details for the moment, our main result can be described as follows. For the precise version, see Theorem 3.6.

THEOREM 1.1. *Consider* (1.1) *for* $f(u) = u - u^3$ *and* $\mu = 0$, *and let* $\varrho > 0$ *be arbitrary, but fixed. If we randomly choose an initial condition* u_o *satisfying*

$$||u_o||_{H^2(\Omega)} \leq C \cdot \varepsilon^k,$$

where $k > 0$ *depends on* ϱ *and* $\dim \Omega$, *then with high probability (independent of* ε*), the solution* u *of* (1.1) *originating at* u_o *will closely follow the solution of the linearized equation as long as*

$$||u(t)||_{H^2(\Omega)} \leq C \cdot \varepsilon^{-1+\varrho+\dim \Omega/4}. \tag{1.8}$$

The above result shows that if a solution of the nonlinear Cahn–Hilliard equation starts sufficiently close to the homogeneous equilibrium $\bar{u}_o \equiv \mu$, then it will almost certainly follow the corresponding solution of the linearized equation up to an unexpectedly large distance from the equilibrium. (The notion of probability is subtle, since this is an infinite-dimensional problem. See the end of subsection 3.4.) Thus, the patterns observed during spinodal decomposition are precisely the patterns generated by the linearized evolution. See Figure 1.1 for an example in two space dimensions. The pictures in this figure are snapshots of the function $v(t)$ at various times t. The shading represents the values of $v(t,x)$, with black and white corresponding to -1 and 1, respectively.

With Theorem 1.1 we partially answer a question raised in our previous paper [31]. There, numerical simulations in one space dimension indicate that the relative distance $||u - v||_{H^2(\Omega)}/||v||_{H^2(\Omega)}$ between the nonlinear solution u and the linear solution v remains bounded by some small ε-independent threshold, as long as the norm of the nonlinear solution is bounded by $C\varepsilon^{-2}$. While our above result does not reproduce the exponent -2, it furnishes a better threshold for the relative distance, namely of the order $O(\varepsilon^{2-\dim \Omega/2})$. This can be seen from our main theorem, Theorem 3.4 of section 3. Leaving out technical details, it can be restated as follows.

THEOREM 1.2. *Again, consider the Cahn–Hilliard equation* (1.1) *with* $f(u) = u - u^3$ *and* $\mu = 0$. *Let* $\varrho > 0$ *be arbitrarily small, but fixed. Let* u_o *denote an initial condition close to* $\bar{u}_o \equiv \mu$, *which is sufficiently close to the subspace of dominating eigenfunctions. Finally, let* u *and* v *be the solutions to* (1.1) *and* (1.2), *respectively, starting at* u_o. *Then as long as*

$$||u(t)||_{H^2(\Omega)} \leq C \cdot \varepsilon^{-1+\varrho+\dim \Omega/4} \cdot ||u_o||^{\varrho}_{H^2(\Omega)} \tag{1.9}$$

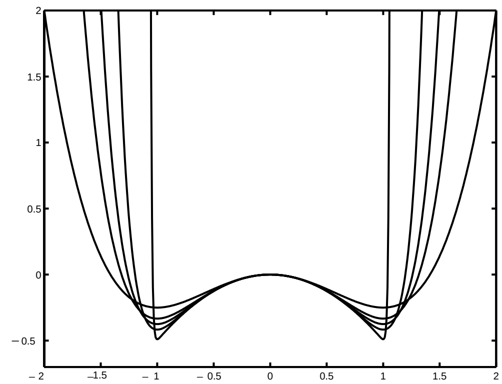

FIG. 1.3. *Double-well potentials F_σ for $\sigma = 2, 4, 6, 10, 100$.*

we have

$$\frac{||u(t) - v(t)||_{H^2(\Omega)}}{||v(t)||_{H^2(\Omega)}} \leq C \cdot \varepsilon^{2-\dim\Omega/2}.$$

In other words, u remains extremely close to v until $||u(t)||_{H^2(\Omega)}$ exceeds the threshold given in (1.9).

Combined with the results of Maier-Paape and Wanner [26, 28], Theorem 1.2 immediately implies Theorem 1.1.

For the sake of simplicity, we consider only the two results above for the special case of (1.1) with $f(u) = u - u^3$ and $\mu = 0$. This can easily be generalized. In fact, by choosing different nonlinearities f, better values for the radii given in (1.8) and (1.9) can be obtained. Consider, for example, the case $f_\sigma(u) = u - u^{1+\sigma}$, where $\sigma \geq 1$. The corresponding double-well potentials F_σ are given by $F_\sigma(u) = u^{2+\sigma}/(2+\sigma) - u^2/2$, as shown for various σ in Figure 1.3. Notice that for $\sigma \to \infty$ these potentials approach the nonsmooth free energy which has been discussed by Blowey and Elliott [4, 5]. We show in section 3 that for $\mu = 0$ and a nonlinearity of this form, the radius in (1.8) can be replaced by

$$C \cdot \varepsilon^{(-2+\dim\Omega/2)\cdot(1-1/\sigma)+\varrho}.$$

A similar statement is valid for the radius given in (1.9). Thus, by choosing a suitable double-well potential F_σ, we can get as close to the order estimate $\varepsilon^{-2+\dim\Omega/2}$ as we wish. Furthermore, the case $\mu \neq 0$ can be reduced to the case $\mu = 0$ by a change of variables, which results in a change of the nonlinearity f. This may, however, lead to a quadratic nonlinearity, i.e., to $\sigma = 1$, and therefore reduce the order of the radius in (1.8).

This paper is organized as follows. Section 2 contains estimates for the relative distance in an abstract setting. After collecting some definitions and assumptions in subsection 2.1, we derive a bound on the absolute distance between solutions of a nonlinear and linear equation in subsection 2.2 originating at the same initial condition u_o. In order to obtain a bound on the relative distance in subsection 2.4, we use a cone condition for the initial condition u_o. This cone condition is presented in subsection 2.3, together with some auxiliary results.

The abstract results are applied to the Cahn–Hilliard equation in section 3. We begin in subsection 3.1 to describe the specific operators that have to be considered.

Furthermore, we present the necessary estimates on the linearized Cahn–Hilliard equation. Sharp estimates on the nonlinearity lie at the heart of our result. These estimates are contained in subsection 3.2. They require a technical condition on the domain Ω, which is, for example, satisfied for generic rectilinear domains. In subsection 3.3 we collect everything to prove our main theorem, Theorem 3.4. The precise version of Theorem 1.1 is formulated and proven in subsection 3.4. Finally, section 4 contains a discussion of our results and points towards further applications and improvements.

2. Results for abstract evolution equations. The following results give precise bounds on how long solutions of a nonlinear equation remain close to solutions of an associated linear equation. For ease of discussion and applicability to other situations, we consider abstract evolution equations. The specific bounds for the Cahn–Hilliard equation are derived in the next section. We rely heavily on the results of Henry [21]. In a different context, his methods have previously been applied to the Cahn–Hilliard equation by Novick-Cohen [30].

2.1. Definitions and assumptions. Let X denote a Hilbert space with scalar product $(\cdot,\cdot)$ and norm $||\cdot||$. Assume that $-A$ is a sectorial operator on X. Then for $\alpha \in (0,1)$ and suitable $a \in \mathbb{R}$ we get the fractional power spaces $X^\alpha = D((-A+aI)^\alpha) \subset X$. These spaces are Hilbert spaces with scalar product $(\cdot,\cdot)_\alpha$ and corresponding norm $||\cdot||_\alpha$; see Henry [21]. We consider evolution equations on these spaces.

For the entire section, we assume the following notation: Let A be as above, and let $F : X^\alpha \to X$ denote a Lipschitz continuous mapping. Then, according to Henry [21, Theorem 3.3.3] and Miklavčič [29], the initial value problem

$$u_t = Au + F(u), \qquad u(0) = u_o \in X^\alpha \tag{2.1}$$

has a unique local solution. Besides this nonlinear evolution equation and its solution u we consider the linear initial value problem

$$v_t = Av\ , \qquad v(0) = u_o \in X^\alpha, \tag{2.2}$$

with the same initial condition u_o. We use the following definition.

Definition 2.1 (existence of solutions). *For some initial condition $u_o \in X^\alpha$ and some (not necessarily maximal) time $T_{\max} > 0$, let $u : [0, T_{\max}] \to X^\alpha$ denote the unique solution to the initial value problem* (2.1)*. Furthermore, we denote the globally defined solution to the linear initial value problem* (2.2) *by $v : [0,\infty) \to X^\alpha$. Note that if $\{S(t) : t \geq 0\}$ denotes the analytic semigroup generated by A, then we have $v(t) = S(t)u_o$ for all $t \geq 0$.*

Our goal is to study by how much the nonlinear solution u differs from the linear solution v. We quantify this using the *relative distance*

$$\frac{||u(t) - v(t)||_\alpha}{||v(t)||_\alpha} \tag{2.3}$$

between the solutions u and v of the nonlinear and linear equations.

For our abstract results to hold we need the following assumptions on the analytic semigroup generated by A. The linearized Cahn–Hilliard equation satisfies this set of assumptions, as is shown in the next section.

ASSUMPTION 2.2 (linear semigroup). *Let $S(t)$ denote the analytic semigroup generated by A. We assume that the estimates*

$$\begin{array}{llllll} ||S(t)\varphi||_\alpha & \leq & K \cdot t^{-\alpha} \cdot e^{\beta t} \cdot ||\varphi|| & \text{for} & t > 0, & \varphi \in X, \quad \text{and} \\ ||S(t)\varphi||_\alpha & \leq & e^{\lambda t} \cdot ||\varphi||_\alpha & \text{for} & t \geq 0, & \varphi \in X^\alpha \end{array}$$

are satisfied for some constants $K \geq 1$, $\beta \in \mathbb{R}$, and $\lambda < \beta$.

Note that the above assumptions are automatically satisfied if A is a self-adjoint operator whose eigenvalues are bounded above by λ.

The last assumption of this subsection is concerned with the nonlinearity F. We assume the following polynomial growth bound along a specific solution u.

ASSUMPTION 2.3 (nonlinearity). *Let $u_o \in X^\alpha$ and let u denote a solution of* (2.1) *as in Definition* 2.1. *We assume that for some constant $M > 0$ and some $\sigma > 0$ we have*

$$||F(u(t))|| \leq M \cdot ||u(t)||_\alpha^{1+\sigma}$$

for all $t \in [0, T_{\max}]$.

2.2. A bound on the absolute distance. The following lemma provides a first estimate on the deviation of the nonlinear and the linear solutions, provided the nonlinearity satisfies a rather restrictive estimate.

LEMMA 2.4. *Consider the initial value problems* (2.1) *and* (2.2) *for some $u_o \in X^\alpha$, and let u and v be given as in Definition* 2.1. *Furthermore, let Assumption* 2.2 *be satisfied and assume that there exist constants $L \geq 0$ and $T \in [0, T_{\max}]$ such that*

$$||F(u(t))|| \leq L||u(t)||_\alpha \qquad \text{for all} \quad t \in [0, T]. \tag{2.4}$$

Then for all $t \in [0, T]$ the absolute distance of u and v satisfies

$$||u(t) - v(t)||_\alpha \;\leq\; \frac{KL \cdot d(\alpha)}{(\beta - \lambda)^{1-\alpha} \cdot (1 - \alpha)} \cdot e^{(\beta+\theta)t} \cdot ||u_o||_\alpha, \tag{2.5}$$

where

$$\theta = (KL \cdot \Gamma(1 - \alpha))^{1/(1-\alpha)}, \tag{2.6}$$

and $d(\alpha)$ depends only on $\alpha \in (0, 1)$. In particular, we have $d(1/2) = 2$.

Proof. Let $w = u - v$. Then w satisfies the integral equation

$$w(t) = \int_0^t S(t - s)F(w(s) + v(s))ds,$$

and the hypotheses of the lemma imply

$$e^{-\beta t}||w(t)||_\alpha \;\leq\; KL \int_0^t (t - s)^{-\alpha} e^{-\beta s} ||w(s)||_\alpha ds + KL||u_o||_\alpha \int_0^t (t - s)^{-\alpha} e^{(\lambda-\beta)s} ds.$$

A straightforward calculation shows that for all $t \geq 0$ we have

$$e^{-t} \cdot \int_0^t s^{-\alpha} e^s ds \;\leq\; \frac{1}{1 - \alpha}.$$

In addition, by changing the variable of integration, one can see that

$$\int_0^t (t-s)^{-\alpha} e^{(\lambda-\beta)s} ds \;=\; (\beta-\lambda)^{\alpha-1} \cdot e^{-(\beta-\lambda)t} \cdot \int_0^{(\beta-\lambda)t} s^{-\alpha} e^s ds.$$

Therefore,

$$e^{-\beta t} ||w(t)||_\alpha \;\le\; M + KL \int_0^t (t-s)^{-\alpha} e^{-\beta s} ||w(s)||_\alpha ds,$$

where $M = KL \cdot ||u_o||_\alpha / ((\beta-\lambda)^{1-\alpha} \cdot (1-\alpha))$. A result of Henry [21, Lemma 7.1.1] finally yields

$$e^{-\beta t} ||w(t)||_\alpha \;\le\; M \cdot E_{1-\alpha}(\theta t),$$

where $\theta = (KL \cdot \Gamma(1-\alpha))^{1/(1-\alpha)}$ and $E_\sigma(x) = \sum_{k=0}^\infty x^{k\sigma}/\Gamma(k\sigma+1)$. Henry also shows that $E_{1-\alpha}(x) \le d(\alpha) e^x$, where $d(\alpha)$ is a constant depending only on α. Particularly, for $\alpha = 1/2$ it is not hard to verify directly that

$$E_{1/2}(x) = e^x \cdot \left(1 + \frac{2}{\sqrt{\pi}} \cdot \int_0^{\sqrt{x}} e^{-s^2} ds \right),$$

so we have $d(1/2) = 2$ as claimed. This completes the proof. ∎

The above lemma is only a first step towards estimating the relative distance between u and v. In order to obtain estimates on the relative distance we also need good lower bounds on the growth of the linear solution v. This calls for introducing certain cone conditions for the initial condition.

2.3. Consequences of the cone condition. In order to derive good lower bounds on the exponential growth of a linear solution $v(t) = S(t)u_o$, we have to make sure that the initial condition u_o possesses a sufficiently nontrivial projection on specific subspaces of X^α, namely, those where $S(t)$ is expanding. For this, we need an additional assumption.

ASSUMPTION 2.5 (splitting of X^α). *Let $S(t)$ and λ be as in Assumption 2.2. Assume that we can decompose the Hilbert space X^α into an orthogonal sum $X^\alpha = X^+ \oplus X^-$ of subspaces which are invariant with respect to $S(t)$. Suppose that X^+ is finite-dimensional. Then we assume further that the estimates*

$$\begin{aligned} &||S(t)\varphi^+||_\alpha \ge e^{\gamma t} \cdot ||\varphi^+||_\alpha \quad \textit{for all} \quad t \ge 0, \qquad \varphi^+ \in X^+, \\ &||S(t)\varphi^-||_\alpha \le e^{\gamma t} \cdot ||\varphi^-||_\alpha \quad \textit{for all} \quad t \ge 0, \qquad \varphi^- \in X^- \end{aligned} \tag{2.7}$$

are satisfied for some constant $0 < \gamma < \lambda$.

The above splitting will normally be induced by a decomposition of the spectrum of the generator A of the semigroup $S(t)$. The spaces X^+ and X^- then correspond to the linear hull of all eigenvectors corresponding to eigenvalues greater than or less than γ, respectively. Note that (2.7) immediately implies

$$||S(t)u_o||_\alpha \ge e^{\gamma t} \cdot ||u_o^+||_\alpha \quad \text{for all} \quad t \ge 0. \tag{2.8}$$

We use this fact by imposing cone conditions on the initial conditions u_o. Let $\delta > 0$. For the splitting $X^\alpha = X^+ \oplus X^-$ introduced in Assumption 2.5 denote the *unstable cone* with opening δ by $\mathcal{C}_\delta$, i.e., define

$$\mathcal{C}_\delta = \left\{ u = u^+ + u^- \in X^+ \oplus X^- = X^\alpha \;:\; ||u^-||_\alpha \le \delta \cdot ||u^+||_\alpha \right\}.$$

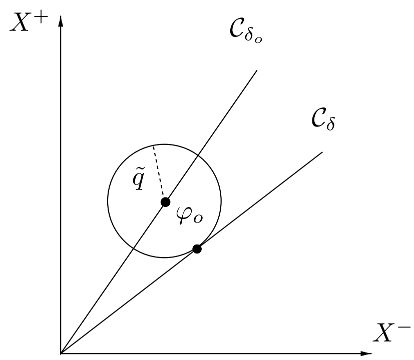

FIG. 2.1. *Points within relative distance q of $\mathcal{C}_{\delta_o}$ must be in $\mathcal{C}_\delta$. Here $\tilde{q} = q||\varphi_o||_\alpha$.*

For nontrivial $u_o \in \mathcal{C}_\delta$, estimate (2.8) implies an exponential lower bound on the growth of the linear solution $S(t)u_o$.

The remainder of this subsection contains two simple consequences of the cone condition which are needed later. The first lemma provides lower and upper bounds on the growth of the nonlinear solution u, assuming that the initial condition u_o is inside a fixed unstable cone and the relative distance between u and the linear solution v can be controlled.

LEMMA 2.6. *Let u and v be given as in Definition* 2.1*, and assume that Assumptions* 2.2 *and* 2.5 *hold. For constants $\delta > 0$ and $q \in (0, 1)$, let $u_o \in \mathcal{C}_\delta$, and suppose that for all $t \in [0, T]$ the relative distance between u and v is at most q, i.e.,*

$$\frac{||u(t) - v(t)||_\alpha}{||v(t)||_\alpha} \leq q. \tag{2.9}$$

Then for every $t \in [0, T]$ the inequality

$$(1 - q) \cdot \frac{1}{\sqrt{1 + \delta^2}} \cdot ||u_o||_\alpha \cdot e^{\gamma t} \;\leq\; ||u(t)||_\alpha \;\leq\; (1 + q) \cdot ||u_o||_\alpha \cdot e^{\lambda t} \tag{2.10}$$

is satisfied.

Proof. Due to $u_o \in \mathcal{C}_\delta$ and the orthogonality of the splitting $X^\alpha = X^+ \oplus X^-$ we get $||u_o^+||_\alpha \geq (1 + \delta^2)^{-1/2} \cdot ||u_o||_\alpha$. Together with (2.7) and (2.9) this yields the first inequality in (2.10). The second one follows from Assumption 2.2 and (2.9). ◻

REMARK 2.7. *Similar to the above proof, the fact that $u_o \in \mathcal{C}_\delta$ implies, with* (2.8)*, the lower bound $||v(t)||_\alpha \geq ||u_o||_\alpha \cdot e^{\gamma t}/\sqrt{1 + \delta^2}$ for all $t \geq 0$.*

The next lemma shows that if we can control the relative distance between two points, one of which is contained in some cone $\mathcal{C}_{\delta_o}$, then the other point will be contained in a somewhat larger cone $\mathcal{C}_\delta$.

LEMMA 2.8. *Let $\delta_o > 0$ and $0 < q < 1/\sqrt{1 + \delta_o^2}$. Define*

$$\delta = \delta_o + \frac{q \cdot \left(1 + \delta_o^2\right)}{\sqrt{1 - q^2} - q \cdot \delta_o}.$$

Then for any $\varphi_o \in \mathcal{C}_{\delta_o}$ and $\varphi \in X^\alpha$ such that $||\varphi - \varphi_o||_\alpha/||\varphi_o||_\alpha \leq q$ we have $\varphi \in \mathcal{C}_\delta$.

Proof. Since we are working with a splitting of a Hilbert space, the result reduces to a fact about planar geometry. See Figure 2.1. ◻

2.4. A bound on the relative distance. In this subsection we combine Lemma 2.4 with the results of the last subsection to obtain precise upper bounds on the relative distance between u and v from Definition 2.1. In order to abbreviate the presentation, we introduce new constants.

DEFINITION 2.9 (introduction of m and N). *With the notation of Definition* 2.1*, Assumptions* 2.2*,* 2.3*, and* 2.5*, and Lemma* 2.4*, let* $N > 0$ *be such that*

$$\frac{K \cdot d(\alpha)}{(\beta - \lambda)^{1-\alpha} \cdot (1-\alpha)} \cdot M \;\leq\; N. \tag{2.11}$$

Moreover, let θ *be defined as in* (2.6) *and let* $m > 0$ *be such that*

$$\beta + \theta - \gamma \;\leq\; m \cdot \gamma. \tag{2.12}$$

Introducing m and N in the above way might seem strange at first sight. However, in our application to the Cahn–Hilliard equation the constants on the left-hand sides of (2.11) and (2.12) will exhibit various dependencies on ε, and it is convenient to incorporate these dependencies into the constants m and N.

Now we have gathered everything to derive our main result of this section. Assume for the moment that all of the above assumptions hold for some initial condition u_o with $T_{\max} = \infty$. Choose $R_1 > 0$ so that $R_1 > ||u_o||_\alpha$, and let

$$T = \sup\left\{\tau > 0 \;:\; ||u(t)||_\alpha < R_1 \text{ for all } 0 \leq t \leq \tau\right\}, \tag{2.13}$$

i.e., let T denote the first exit time of the nonlinear solution u from the ball $B_{R_1}(0)$. We want to estimate the relative distance (2.3) between $u(t)$ and $v(t)$ for $t \in [0, T]$. Due to Assumption 2.3 and (2.13) we can bound the Lipschitz constant L in (2.4) by $M \cdot R_1^\sigma$. Then combining (2.5) with (2.11) and (2.12) shows that for all $t \in [0, T]$ we have

$$\begin{aligned} ||u(t) - v(t)||_\alpha &\leq \frac{KL \cdot d(\alpha)}{(\beta-\lambda)^{1-\alpha} \cdot (1-\alpha)} \cdot e^{(\beta+\theta)t} \cdot ||u_o||_\alpha \\ &\leq N \cdot R_1^\sigma \cdot e^{m\gamma t} \cdot e^{\gamma t} \cdot ||u_o||_\alpha. \end{aligned} \tag{2.14}$$

From Remark 2.7, we see that to bound the relative distance between u and v starting in a cone, we need an estimate of the form

$$||u(t) - v(t)||_\alpha \leq \zeta \cdot e^{\gamma t} \cdot ||u_o||_\alpha, \tag{2.15}$$

where $\zeta > 0$ is some small constant. Due to (2.14) this can be achieved if we choose R_1 small enough. However, in order to bound the exponential term $e^{m\gamma t} \leq e^{m\gamma T}$ we need an upper bound on the time T defined in (2.13). This is closely tied to obtaining lower bounds on the growth of u, which in turn leads to a cone condition on the initial condition u_o.

All of this is put together in the following theorem, resulting in an estimate of the form (2.15), as well as a bound on the relative distance between u and v. In particular, the result gives the maximal value for the end radius R_1 in terms of ζ.

THEOREM 2.10. *Let* u *and* v *be given as in Definition* 2.1 *and* m *and* N *be as in Definition* 2.9*, and suppose that Assumptions* 2.2*,* 2.3*, and* 2.5 *are satisfied. Fix two positive constants* $\delta > 0$ *and* $\zeta \in (0, 1/2)$*, choose a constant* $R_o > 0$ *with*

$$R_o < \left(\frac{\zeta}{N}\right)^{1/\sigma} \cdot 2^{-m/\sigma} \cdot \left(1 + \delta^2\right)^{-(m+1)/(2\sigma)}, \tag{2.16}$$

and define

$$R_1 = R_o^{m/(m+\sigma)} \cdot \left(\frac{\zeta}{N}\right)^{1/(m+\sigma)} \cdot 2^{-m/(m+\sigma)} \cdot \left(1+\delta^2\right)^{-(m+1)/(2m+2\sigma)} . \tag{2.17}$$

Finally, assume that the initial condition satisfies $u_o \in \mathcal{C}_\delta$ *and* $R_o \le ||u_o||_\alpha < R_1$, *and define*

$$T = \sup\{\tau \in [0, T_{\max}] \; : \; ||u(t)||_\alpha < R_1 \text{ for all } 0 \le t \le \tau\} . \tag{2.18}$$

Then for all $t \in [0,T]$ *we have*

$$\frac{||u(t) - v(t)||_\alpha}{||v(t)||_\alpha} \le \zeta . \tag{2.19}$$

REMARK 2.11. *If we define* R_1 *as in* (2.17)*, then* $R_1 > R_o$ *is satisfied if and only if* (2.16) *holds.*

Proof. Let $T_0 \in (0, T_{\max}]$ be the maximal time such that for all $0 \le t < T_0$ the relative distance $||u(t) - v(t)||_\alpha / ||v(t)||_\alpha$ is strictly less than $1/2$. Then Lemma 2.6 implies

$$||u(t)||_\alpha \ge \frac{1}{2\sqrt{1+\delta^2}} \cdot e^{\gamma t} \cdot ||u_o||_\alpha \quad \text{for all} \quad 0 \le t \le T_0.$$

Now choose $T_1 \in (0, T_0]$ maximal so that for all $t \in [0, T_1)$ we have $||u(t)||_\alpha < R_1$. Together with $||u_o||_\alpha \ge R_o$ the above inequality then yields

$$e^{\gamma t} \le 2\sqrt{1+\delta^2} \cdot \frac{R_1}{R_o} \quad \text{for all} \quad 0 \le t \le T_1.$$

Combining this estimate with (2.14), we obtain

$$\begin{aligned} ||u(t) - v(t)||_\alpha &\le N \cdot R_1^\sigma \cdot e^{m\gamma t} \cdot e^{\gamma t} \cdot ||u_o||_\alpha \\ &\le N \cdot R_1^\sigma \cdot \left(2\sqrt{1+\delta^2}\right)^m \cdot \frac{R_1^m}{R_o^m} \cdot e^{\gamma t} \cdot ||u_o||_\alpha \end{aligned}$$

for all $0 \le t \le T_1$. Finally, the definition (2.17) of R_1 implies

$$||u(t) - v(t)||_\alpha \le \frac{\zeta}{\sqrt{1+\delta^2}} \cdot e^{\gamma t} \cdot ||u_o||_\alpha \quad \text{for all} \quad 0 \le t \le T_1.$$

Combining the above with Remark 2.7 proves (2.19) for all $t \in [0, T_1]$.

In order to finish the proof of Theorem 2.10 we have only to verify $T_1 = T$. This is obvious if $T_0 = T_{\max}$. If $T_0 < T_{\max}$, then by the continuity of u and v we know that the relative distance between u and v at time T_0 is exactly $1/2$. On the other hand, we already showed that at time T_1 this relative distance is at most $\zeta < 1/2$. Therefore, $T_1 < T_0$ and $||u(T_1)||_\alpha = R_1$, i.e., $T_1 = T$ for $T_0 < T_{\max}$ as well. ∎

REMARK 2.12. *In the above proof we applied Lemma* 2.4 *once with* $L = M \cdot R_1^\sigma$, *and this resulted in the maximal radius* R_1 *given by* (2.17)*. However, the nonlinearity* F *varies polynomially with* $||u||_\alpha$*. It therefore seems plausible that applying Lemma* 2.4 *several times with growing values of* L *would improve the results. Although there are several technical issues that have to be addressed in such an iterative method, it can be done. The main difference in the resulting improved radius* R_1 *is that the exponent* $1/(m+\sigma)$ *of* ζ/N *can be replaced by* $1/\sigma$*. While this might be important in certain applications, it provides only a negligible improvement for the Cahn–Hilliard equation, as in this case the constant* $m > 0$ *can be chosen arbitrarily small. We have decided not to present the iterative proof, since it would come at the cost of simplicity.*

3. The Cahn–Hilliard equation. In this section, we apply the abstract results to the Cahn–Hilliard equation in dimension $n = 1$, 2, and 3, linearized at the homogeneous equilibrium solution $\bar{u}_o \equiv \mu$. We assume that $-f$ is the derivative of a sufficiently smooth double-well potential F and that μ lies in the spinodal interval, i.e., we assume $f'(\mu) > 0$. As we pointed out in the introduction, the Cahn–Hilliard equation (1.1) generates a nonlinear semiflow in the space $\mu + X^{1/2}$, where $X^{1/2}$ is defined in (1.7).

In order to simplify the presentation, we perform a change of variables and consider the equation

$$\begin{aligned} u_t &= -\Delta(\varepsilon^2 \Delta u + f(\mu + u)) \quad \text{in} \quad \Omega, \\ &\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \\ &\int_\Omega u dx = 0; \end{aligned} \tag{3.1}$$

see Maier-Paape and Wanner [28, section 3.1], Novick-Cohen [30], and Zheng [32]. For any solution u of (3.1), the sum $\mu + u$ solves the original Cahn–Hilliard equation (1.1) in the affine space $\mu + X^{1/2}$. In other words, by redefining the nonlinearity f in a suitable way, we may assume $\mu = 0$ without loss of generality and we do this from now on. Thus, we apply the abstract results of the last section to (3.1), which generates a semiflow in the Hilbert space $X^{1/2}$. Furthermore, we consider only nonlinearities f of the form $f(u) = u - u^{1+\sigma}$ for an even integer $\sigma \geq 2$. This can easily be generalized to other nonlinearities.

3.1. Abstract setting and linear estimates. Let X be defined as in (1.3). We consider the linear operator $A_\varepsilon : X \to X$ given by

$$A_\varepsilon u = -\Delta(\varepsilon^2 \Delta u + f'(\mu)u),$$

with domain $D(A_\varepsilon)$ as in (1.4). The corresponding fractional power space for $\alpha = 1/2$ is given by the Hilbert space $X^{1/2}$ from (1.7), equipped with a norm $|| \cdot ||_{1/2}$. In the following, instead of $|| \cdot ||_{1/2}$ we use the norm $||u||_* = (||u||_{L^2}^2 + ||\Delta u||_{L^2}^2)^{1/2}$, which is equivalent to both $|| \cdot ||_{1/2}$ and the $H^2(\Omega)$-norm; see Maier-Paape and Wanner [28].

The following lemma shows that the analytic semigroup $S_\varepsilon(t)$ generated by the linear operator A_ε defined above satisfies all the conditions of our abstract result. In particular, it gives the exact ε-dependence of the constants.

LEMMA 3.1. *Let A_ε, X, and $X^{1/2}$ be defined as above, and let $S_\varepsilon(t)$ denote the analytic semigroup generated by A_ε. Let $\alpha = 1/2$ and $\lambda_\varepsilon^{\max} = f'(\mu)^2/(4\varepsilon^2)$ from* (1.6). *Finally, choose an arbitrary $\beta_o > 0$. Then all the estimates in Assumption* 2.2 *are satisfied if we choose $\lambda = \lambda_\varepsilon^{\max}$,*

$$\beta = (1 + \beta_o) \cdot \lambda_\varepsilon^{\max},$$

and

$$K = \frac{1}{\varepsilon} \cdot \sqrt{\frac{1 + \beta_o + 4\varepsilon^4/f'(\mu)^2}{2e \cdot \beta_o}}.$$

Proof. By Lemma 3.4 in [28, p. 210], we have $||S_\varepsilon(t)u||_* \leq e^{\lambda t}||u||_*$, with λ as above. Furthermore, for any $\beta > \lambda$, Lemma 3.4 in [28, p. 210] implies that

$$||S_\varepsilon(t)u||_* \leq K \cdot t^{-1/2} \cdot e^{\beta t} \cdot ||u||,$$

as long as

$$K \geq \sup_{s \geq 0} \sqrt{\frac{1+s^2}{2e(\beta - f'(\mu)s + \varepsilon^2 s^2)}}.$$

A tedious but straightforward calculation shows that this inequality is satisfied for the K and β specified in the statement of this lemma. □

3.2. Growth estimates for the nonlinearity. Next we have to verify the assumptions on the nonlinearity $F : X^{1/2} \to X$, which is defined as

$$F(u) = -\Delta(g(u)), \qquad \text{where} \quad g(u) = f(\mu + u) - f'(\mu)u. \tag{3.2}$$

In order to achieve the desired sharp estimates we have to restrict our attention to a specific part of the phase space, defined in terms of the eigenfunctions of A_ε. Furthermore, we need a condition on the domain Ω in order for our results to hold.

Recall from the introduction that the operator $-\Delta : X \to X$ subject to Neumann boundary conditions has a complete $L^2(\Omega)$-orthonormal set of eigenfunctions ψ_1, ψ_2, ψ_3, ..., with corresponding eigenvalues $0 < \kappa_1 \leq \kappa_2 \leq \kappa_3 \leq \cdots \to \infty$. We need the following assumptions on the eigenfunctions ψ_k and eigenvalues κ_k, which is in fact an assumption on the domain Ω.

ASSUMPTION 3.2 (properties of the domain Ω). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, where $n \in \{1, 2, 3\}$. We assume that the eigenvalues κ_k satisfy*

$$\kappa_k \sim k^{2/n} \quad \text{as} \quad k \to \infty. \tag{3.3}$$

Moreover, assume that there are positive constants C_1 and C_2 such that the following $L^\infty(\Omega)$-estimates hold: for all $k \in \mathbb{N}$ we have both

$$||\psi_k||_{L^\infty} \leq C_1 \tag{3.4}$$

and

$$||\nabla\psi_k||_{L^\infty} \leq C_2 \cdot \sqrt{\kappa_k}. \tag{3.5}$$

In order to simplify notation we write $||\nabla\psi_k||_{L^\infty}$ instead of $||\,|\nabla\psi_k|\,||_{L^\infty}$. Furthermore, recall that the eigenfunctions ψ_k are normalized with respect to the $L^2(\Omega)$-norm.

The above assumptions are automatically satisfied for one-dimensional domains. Also, (3.3) holds for $n = 2$ and $n = 3$ if all the eigenvalues of $-\Delta$ are simple. This is an immediate consequence of the asymptotic distribution of the eigenvalues of $-\Delta$; see, for example, Courant and Hilbert [12, p. 442] or Edmunds and Evans [14]. Since it can easily be verified that both (3.4) and (3.5) are true for rectangular domains, Assumption 3.2 is therefore satisfied for generic rectangular domains. Although we do not know of any more general geometric condition on Ω which implies our above assumptions, these assumptions are also frequently used elsewhere; see, for example, Da Prato and Zabczyk [13, p. 139].

As mentioned above, we can only obtain useful estimates on the growth of the nonlinearity $F(u)$ in portions of the phase space $X^{1/2}$. For the sake of simplicity of our presentation, we chose this region to be a cone around a dominating subspace of Maier-Paape and Wanner [26]. This already considerably improves known results while at the same time rendering a technically simple discussion. Recall that if we define

$$\tilde{\varphi}_k = \frac{1}{\sqrt{1+\kappa_k^2}} \cdot \psi_k \quad \text{for} \quad k \in \mathbb{N},$$

then the $\tilde{\varphi}_k$ form a complete orthonormal set of eigenfunctions of A_ε with respect to the scalar product induced by $||\cdot||_*$. The corresponding eigenvalues $\tilde{\lambda}_{k,\varepsilon}$ are given by (1.5). Fix some constant $\gamma_o \in (0,1)$ and let

$$X_\varepsilon^+ = \text{span } \left\{ \tilde{\varphi}_k \ : \ \tilde{\lambda}_{k,\varepsilon} \geq \gamma_o \cdot \lambda_\varepsilon^{\max} \right\} \ \subset \ X^{1/2}, \tag{3.6}$$

$$X_\varepsilon^- = \text{span } \left\{ \tilde{\varphi}_k \ : \ \tilde{\lambda}_{k,\varepsilon} < \gamma_o \cdot \lambda_\varepsilon^{\max} \right\} \ \subset \ X^{1/2}. \tag{3.7}$$

See Figure 1.2. Then X_ε^+ is a dominating subspace with dimension proportional to ε^{-n} asymptotically as ε goes to zero; see Maier-Paape and Wanner [26, p. 442]. We consider cones $\mathcal{K}_\delta \subset X^{1/2}$ with respect to the decomposition $X^{1/2} = X_\varepsilon^+ \oplus X_\varepsilon^-$ defined as

$$\mathcal{K}_\delta = \left\{ u \in X^{1/2} \ : \ ||u^-||_* \leq \delta \cdot ||u^+||_*, \quad u = u^+ + u^- \in X_\varepsilon^+ \oplus X_\varepsilon^- \right\},$$

for some $\delta > 0$. The following lemma contains the desired estimate on the $L^2(\Omega)$-norm of the nonlinearity $F(u) = -\Delta(f(u) - f'(\mu)u) = -\Delta(g(u))$; see (3.2).

Lemma 3.3. *Let Ω be such that Assumption 3.2 is satisfied, and consider a function $g : \mathbb{R} \to \mathbb{R}$ given by $g(u) = u^{1+\sigma} \cdot \tilde{g}(u)$, where $\sigma \geq 1$ and $\tilde{g}$ is C^2 on an open interval containing 0. Finally, define $F(u) = -\Delta(g(u))$, let $\delta_o > 0$ be arbitrary, and set*

$$\delta_\varepsilon = \delta_o \cdot \varepsilon^{2-n/2}. \tag{3.8}$$

Then there exist ε-independent positive constants M_1 and M_2, such that for every $\varepsilon \in (0,1)$ and every function $u \in \mathcal{K}_{\delta_\varepsilon}$ with

$$||u||_* \leq M_1 \cdot \varepsilon^{-2+n/2} \tag{3.9}$$

we have

$$||F(u)||_{L^2} \leq M_2 \cdot \varepsilon^{(2-n/2)\cdot\sigma} \cdot ||u||_*^{\sigma+1}. \tag{3.10}$$

The constants M_1 and M_2 depend only on g, δ_o, and Ω.

Proof. Since $\tilde{g}$ is C^2 on an open interval containing 0 there exist constants $\tilde{M}_1 > 0$ and $\tilde{M}_2 > 0$ such that

$$|g'(s)| \leq \tilde{M}_1 \cdot |s|^\sigma \quad \text{and} \quad |g''(s)| \leq \tilde{M}_1 \cdot |s|^{\sigma-1} \quad \text{for all} \quad |s| \leq \tilde{M}_2. \tag{3.11}$$

We show below that for δ_ε given in (3.8) there exist numbers $\tilde{M}_3$ and $\tilde{M}_4$ such that for arbitrary $\varepsilon \in (0,1)$ and every $u \in \mathcal{K}_{\delta_\varepsilon}$ the estimates

$$||u||_{L^\infty} \leq \tilde{M}_3 \cdot \varepsilon^{2-n/2} \cdot ||u||_* \qquad \text{and} \tag{3.12}$$

$$||\nabla u||_{L^4} \leq \tilde{M}_4 \cdot \varepsilon^{1-n/4} \cdot ||u||_* \tag{3.13}$$

hold. (In order to simplify notation we write $||\nabla u||_{L^4}$ instead of $||\,|\nabla u|\,||_{L^4}$.)

Let $\varepsilon \in (0,1)$ and $M_1 = \tilde{M}_2/\tilde{M}_3$. Choose $u \in \mathcal{K}_{\delta_\varepsilon}$ satisfying (3.9). Then (3.12) implies $||u||_{L^\infty} \leq \tilde{M}_2$, and (3.11) yields

$$||g'(u)||_{L^\infty} \leq \tilde{M}_1 \cdot ||u||_{L^\infty}^\sigma \qquad \text{and} \qquad ||g''(u)||_{L^\infty} \leq \tilde{M}_1 \cdot ||u||_{L^\infty}^{\sigma-1}.$$

Together with $F(u) = -g'(u)\Delta u - g''(u)|\nabla u|^2$, (3.12), and (3.13), this implies

$$\begin{aligned} ||F(u)||_{L^2} &\leq ||g'(u)||_{L^\infty} \cdot ||\Delta u||_{L^2} + ||g''(u)||_{L^\infty} \cdot ||\nabla u||_{L^4}^2 \\ &\leq \tilde{M}_1 \tilde{M}_3^\sigma \cdot \varepsilon^{(2-n/2)\cdot\sigma} \cdot ||u||_*^{\sigma+1} + \tilde{M}_1 \tilde{M}_3^{\sigma-1} \tilde{M}_4^2 \cdot \varepsilon^{(2-n/2)\cdot\sigma} \cdot ||u||_*^{\sigma+1}, \end{aligned}$$

which is the desired estimate (3.10).

It remains to prove (3.12) and (3.13). To that end, we have to use a further splitting of X_ε^-. Namely, since the eigenvalues κ_k of the negative Laplacian are increasing, we can choose k_s to be the smallest integer such that $\tilde{\varphi}_{k_s} \in X_\varepsilon^+$. In addition, let k_l be the largest integer such that $\tilde{\varphi}_{k_l} \in X_\varepsilon^+$. Now we decompose X_ε^- into the space $X_\varepsilon^\flat$, spanned by $\{\tilde{\varphi}_k : k < k_s\}$ and the space $X_\varepsilon^\sharp$, spanned by $\{\tilde{\varphi}_k : k > k_l\}$.

(a) *A bound on* $||u||_{L^\infty}$ *for arbitrary* $u \in \mathcal{K}_{\delta_\varepsilon}$. Due to the continuity of the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$, it suffices to prove (3.12) for functions of the form $u = \sum_{k=k_1}^{k_2} \alpha_k \tilde{\varphi}_k$, for arbitrary integers $0 < k_1 \le k_2$, since functions of this type are dense in $X^{1/2}$. Then Hölder's inequality and (3.4) immediately yield

$$(3.14)\quad ||u||_{L^\infty} \le \sum_{k=k_1}^{k_2} |\alpha_k| \cdot ||\tilde{\varphi}_k||_{L^\infty} \le C_1 \cdot \underbrace{\left(\sum_{k=k_1}^{k_2} \alpha_k^2 \right)^{1/2}}_{=||u||_*} \cdot \left(\sum_{k=k_1}^{k_2} \frac{1}{1+\kappa_k^2} \right)^{1/2} .$$

We begin with considering the two cases $u \in X_\varepsilon^+ \oplus X_\varepsilon^\sharp$ and $u \in X_\varepsilon^\flat$ separately.

If $u \in X_\varepsilon^+ \oplus X_\varepsilon^\sharp$ we have $k_1 = k_s$. Moreover, according to Maier-Paape and Wanner [26] the constant k_1 is proportional to ε^{-n} as $\varepsilon \to 0$. Employing (3.3) this furnishes

$$\begin{aligned} \sum_{k=k_1}^{k_2} \frac{1}{1+\kappa_k^2} &\le C \cdot \sum_{k=k_1}^{k_2} k^{-4/n} \le C \cdot \int_{k_1-1}^{k_2} \tau^{-4/n} d\tau \\ &\le C \cdot (k_s - 1)^{1-4/n} \le C \cdot \varepsilon^{4-n} . \end{aligned}$$

Note that here and in what follows, in order to simplify notation C is taken to mean a constant independent of ε, though not always the same constant. Together with (3.14) this implies

$$(3.15)\qquad ||u||_{L^\infty} \le C \cdot \varepsilon^{2-n/2} \cdot ||u||_* \qquad \text{for all} \quad u \in X_\varepsilon^+ \oplus X_\varepsilon^\sharp .$$

Now consider $u \in X_\varepsilon^\flat$, i.e., assume $k_1 = 1$ and $k_2 = k_s - 1$ in (3.14). Due to Sobolev's embedding theorem there exists a constant C which depends only on the domain Ω such that

$$(3.16)\qquad ||\Upsilon||_{L^\infty} \le C||\Upsilon||_* \qquad \text{for all} \quad \Upsilon \in H^2(\Omega).$$

Finally, let $u \in \mathcal{K}_{\delta_\varepsilon}$ be arbitrary with δ_ε as in (3.8). Then we can write $u = u^+ + u^\sharp + u^\flat \in X_\varepsilon^+ \oplus X_\varepsilon^\sharp \oplus X_\varepsilon^\flat$, and

$$||u^\flat||_* \le ||u^\flat + u^\sharp||_* \le \delta_\varepsilon \cdot ||u^+||_* \le \delta_o \cdot \varepsilon^{2-n/2} \cdot ||u||_*.$$

Together with (3.15) and (3.16) this readily implies (3.12).

(b) *A bound on* $||\nabla u||_{L^4}$ *for arbitrary* $u \in \mathcal{K}_{\delta_\varepsilon}$. Let $u \in \mathcal{K}_{\delta_\varepsilon}$ be arbitrary, and assume $u = u^+ + u^- \in X_\varepsilon^+ \oplus X_\varepsilon^-$. Due to the Sobolev embedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$ (see Adams [1]) there exists a constant C which depends only on the domain Ω such that

$$||\nabla\Upsilon||_{L^4} \le C||\Upsilon||_* \qquad \text{for all} \quad \Upsilon \in H^2(\Omega).$$

Because of $u \in \mathcal{K}_{\delta_\varepsilon}$ and $\varepsilon \in (0,1)$ we further deduce

$$(3.17)\qquad ||\nabla u^-||_{L^4} \le C \cdot ||u^-||_* \le C \cdot \varepsilon^{2-n/2} \cdot ||u^+||_* \le C \cdot \varepsilon^{1-n/4} \cdot ||u^+||_*.$$

Now consider $u^+ \in X_\varepsilon^+$. We can write this as $u^+ = \sum_{k=k_s}^{k_l} \alpha_k \tilde{\varphi}_k$. Due to the Neumann boundary condition of the $\tilde{\varphi}_k$, integration by parts yields

$$||\nabla u^+||_{L^2}^2 = (-\Delta u^+, u^+)_{L^2} = \sum_{k=k_s}^{k_l} \kappa_k \alpha_k^2 ||\tilde{\varphi}_k||_{L^2}^2 = \sum_{k=k_s}^{k_l} \alpha_k^2 \cdot \frac{\kappa_k}{1+\kappa_k^2}. \tag{3.18}$$

According to Maier-Paape and Wanner [26] the index k_s is proportional to ε^{-n}, and together with assumption (3.3) we get

$$\frac{\kappa_k}{1+\kappa_k^2} \le \kappa_{k_s}^{-1} \le C\varepsilon^2 \qquad \text{for all} \quad k = k_s, \ldots, k_l.$$

Thus (3.18) implies

$$||\nabla u^+||_{L^2} \le C \cdot \varepsilon \cdot \left(\sum_{k=k_s}^{k_l} \alpha_k^2 \right)^{1/2} = C \cdot \varepsilon \cdot ||u^+||_*. \tag{3.19}$$

Next we want to obtain an estimate on the $L^\infty(\Omega)$-norm of $|\nabla u^+|$. Using Hölder's inequality and (3.5) we get

$$\begin{aligned} ||\nabla u^+||_{L^\infty} &\le \sum_{k=k_s}^{k_l} |\alpha_k| \cdot ||\nabla \tilde{\varphi}_k||_{L^\infty} \le C_2 \cdot \sum_{k=k_s}^{k_l} |\alpha_k| \cdot \frac{\sqrt{\kappa_k}}{\sqrt{1+\kappa_k^2}} \\ &\le C_2 \cdot ||u^+||_* \cdot \left(\sum_{k=k_s}^{k_l} \frac{\kappa_k}{1+\kappa_k^2} \right)^{1/2} . \end{aligned}$$

Since both k_s and k_l are proportional to ε^{-n} we deduce with (3.3) the estimate

$$\sum_{k=k_s}^{k_l} \frac{\kappa_k}{1+\kappa_k^2} \le \frac{k_l - k_s + 1}{\kappa_{k_s}} \le C \cdot \varepsilon^{2-n},$$

and therefore

$$||\nabla u^+||_{L^\infty} \le C \cdot \varepsilon^{1-n/2} \cdot ||u^+||_*.$$

Together with (3.19) the last estimate yields

$$||\nabla u^+||_{L^4}^4 = \int_\Omega |\nabla u^+|^4 dx \le ||\nabla u^+||_{L^\infty}^2 \cdot ||\nabla u^+||_{L^2}^2 \le C \cdot \varepsilon^{4-n} \cdot ||u^+||_*^4.$$

In view of (3.17) this completes the proof of (b). □

3.3. Unexpectedly linear behavior. We are finally in a position to put everything together. The following main theorem of our paper gives conditions on the initial condition u_o of the nonlinear Cahn–Hilliard equation (1.1) which imply that its solution u originating at u_o closely follows the corresponding solution of the linearized equation (1.2) up to an unexpectedly large distance from the equilibrium. For a detailed discussion as to why this is unexpected, see [31]. In view of the fact that the estimates on the nonlinearity F derived in the last subsection hold only in certain unstable cones around a dominating subspace X_ε^+, we have to assume that the initial condition u_o is contained in such a cone. Recall that the definition of X_ε^+ given in (3.6) depends on a constant $\gamma_o \in (0,1)$, which determines the cut-off for the dominating eigenvalues.

THEOREM 3.4. *Consider the Cahn–Hilliard equation* (1.1) *with* $f(u) = u - u^{1+\sigma}$ *for some* $\sigma \geq 1$, *let* $\mu = 0$, *and suppose that Assumption* 3.2 *holds. Furthermore, choose and fix constants* $\delta_o \in (0,1/2)$ *and* $\varrho \in (0,1)$.

Then there exist constants $D > 0$ *and* $\gamma_o = \gamma_o(\varrho, \sigma, n) \in (0,1)$ *such that for the splitting defined in* (3.6), (3.7) *and arbitrary* $\varepsilon \in (0,1)$, *the following holds. If* $u_o \in \mathcal{K}_{\delta_\varepsilon}$ *with* $\delta_\varepsilon = \delta_o \cdot \varepsilon^{2-n/2}$ *is any initial condition satisfying*

$$0 < ||u_o||_* \leq \min\left\{1, \ \left(D \cdot \varepsilon^{(-2+n/2)\cdot(1-1/\sigma)+\varrho}\right)^{1/(1-\varrho)}\right\}, \tag{3.20}$$

and if u *and* v *denote the solutions of* (1.1) *and* (1.2), *respectively, then there exists a first time* $T > 0$ *such that*

$$||u(T)||_* = D \cdot \varepsilon^{(-2+n/2)\cdot(1-1/\sigma)+\varrho} \cdot ||u_o||_*^\varrho, \tag{3.21}$$

and for all $t \in [0,T]$ *we have*

$$\frac{||u(t) - v(t)||_*}{||v(t)||_*} \leq \frac{\delta_o}{2} \cdot \varepsilon^{2-n/2}. \tag{3.22}$$

Proof. Choose $m > 0$ constant such that both

$$\frac{m}{m+\sigma} \leq \varrho \quad \text{and} \quad \left(-2 + \frac{n}{2}\right) \cdot \frac{\sigma - 1}{\sigma + m} \leq \left(-2 + \frac{n}{2}\right) \cdot \left(1 - \frac{1}{\sigma}\right) + \varrho \tag{3.23}$$

are satisfied, and fix γ_o such that

$$\frac{1}{m+1} < \gamma_o < 1$$

holds. Let $\alpha = 1/2$, $\gamma = \gamma_o \cdot \lambda_\varepsilon^{\max}$, $\beta_o = \gamma_o \cdot (m + 1 - 1/\gamma_o)/2 > 0$, and define λ, β, K as in Lemma 3.1. Then there is an ε-independent constant N_o such that for all $\varepsilon \in (0,1)$ we have

$$\frac{K \cdot d(\alpha)}{(\beta - \lambda)^{1-\alpha} \cdot (1-\alpha)} \leq N_o.$$

Set

$$N = N_o \cdot M_2 \cdot \varepsilon^{(2-n/2)\cdot\sigma}, \qquad M = M_2 \cdot \varepsilon^{(2-n/2)\cdot\sigma},$$

with M_2 as in Lemma 3.3, but for the larger cone $\mathcal{K}_{2\delta_\varepsilon}$. Choose $T_{\max} > 0$ maximal such that for all $t \in [0, T_{\max})$ we have $u(t) \in \mathcal{K}_{2\delta_\varepsilon}$ and

$$||u(t)||_* < \left(\frac{(\beta_o \cdot \lambda_\varepsilon^{\max})^{1-\alpha}}{K \cdot M \cdot \Gamma(1-\alpha)}\right)^{1/\sigma}. \tag{3.24}$$

Notice that the right-hand side of (3.24) is proportional to $\varepsilon^{-2+n/2}$ for $\varepsilon \to 0$. Let the spaces X^+ and X^- be defined as in (3.6) and (3.7), respectively.

With the above definitions it is straightforward to verify that Assumptions 2.2, 2.3, and 2.5 are satisfied. Finally, choose the constant $D > 0$ in such a way that the expression $D \cdot \varepsilon^{(-2+n/2)\cdot(1-1/\sigma)+\varrho}$ is bounded above by both the right-hand side in (3.24) and by $(\delta_\varepsilon/(2N))^{1/(m+\sigma)} \cdot 2^{-m/(m+\sigma)} \cdot (1+\delta_\varepsilon^2)^{-(m+1)/(2m+2\sigma)}$, for $\varepsilon \in (0,1)$. This is possible due to (3.23). Let $\zeta = \delta_\varepsilon/2$, and let $u_o \in \mathcal{K}_{\delta_\varepsilon}$ satisfy (3.20). Define

$$R_1 = ||u_o||_*^{m/(m+\sigma)} \cdot \left(\frac{\delta_\varepsilon}{2N}\right)^{1/(m+\sigma)} \cdot 2^{-m/(m+\sigma)} \cdot \left(1+\delta_\varepsilon^2\right)^{-(m+1)/(2m+2\sigma)},$$

and let $T_1 \in [0, T_{\max}]$ be defined as in (2.18). Due to our choice of u_o and D we have

$$R_1 \geq R = D \cdot \varepsilon^{(-2+n/2)\cdot(1-1/\sigma)+\varrho} \cdot ||u_o||_*^{\varrho}.$$

Let $T_o \in [0, T_1]$ denote the maximal time such that $||u(t)||_* < R$ for all $t \in [0, T_o)$. Then by Theorem 2.10, (3.22) holds for all $t \in [0, T_o]$, and in order to finish the proof of the theorem we have only to verify $||u(T_o)||_* = R$.

Since this is obviously satisfied if $T_o < T_{\max}$, let us assume $T_o = T_{\max}$ and $||u(T_o)||_* < R$, and arrive at a contradiction. Due to our choice of D and $||u_o||_* \leq 1$, the norm $||u(T_o)||_*$ is strictly less than the right-hand side of (3.24); the definition of $T_{\max}$ shows that $u(T_o)$ has to lie on the boundary of the cone $\mathcal{K}_{2\delta_\varepsilon}$. On the other hand, the estimate (3.22) is satisfied for all $t \in [0, T_{\max}]$. Since the cone $\mathcal{K}_{\delta_\varepsilon}$ is positively invariant for linear solutions, $v(T_o) \in \mathcal{K}_{\delta_\varepsilon}$. Lemma 2.8 shows that $u(T_o) \in \mathcal{K}_{c\cdot\delta_\varepsilon}$ for some $c < 2$, i.e., it cannot be contained in the boundary of $\mathcal{K}_{2\delta_\varepsilon}$. Thus we have a contradiction, and $T_o < T_{\max}$. This completes the proof. ∎

REMARK 3.5 (ε-dependence of the time T). *If the initial conditions u_o are chosen in such a way that $||u_o||_*$ depends polynomially on ε, then it is not hard to show that the time T in* (3.21) *is proportional to $\varepsilon^2 \cdot |\ln \varepsilon|$ as $\varepsilon \to 0$.*

3.4. Entering the cone. As it stands, our main result does not make any statement about how likely it is to find an appropriate initial condition u_o. The following theorem is a corollary to Theorem 3.4 showing that most solutions starting near the equilibrium solution enter the regime of linearity.

THEOREM 3.6. *Consider the Cahn–Hilliard equation* (1.1) *with $f(u) = u - u^{1+\sigma}$ for some $\sigma \geq 1$, let $\mu = 0$, and suppose that Assumption* 3.2 *holds. Furthermore, choose and fix constants $\delta_o \in (0, 1/2)$ and $\varrho \in (0,1)$. Then there exists $k > 0$ such that with high probability (independent of ε), solutions of* (1.1) *starting at an initial condition u_o satisfying*

$$||u_o||_* \leq C \cdot \varepsilon^k$$

enter the region of unexpectedly linear behavior as in Theorem 3.4. *The value of k depends on ϱ. Namely, $\varrho \to 0$ implies $k \to \infty$.*

Proof. This follows immediately from Theorem 3.4 above and the results in Maier-Paape and Wanner [28]. Notice that the latter results have to be slightly modified in order to treat the ε-dependent cone opening. See [28, Corollary 2.1]. ∎

Similar to Remark 3.5, one can easily show that if the initial conditions u_o in Theorem 3.6 are chosen in such a way that $||u_o||_*$ depends polynomially on ε, then the time it takes the solutions to enter the region of unexpectedly linear behavior is again proportional to $\varepsilon^2 \cdot |\ln \varepsilon|$.

Let us close this section with some remarks on the notion of high probability mentioned in the above theorem. It was pointed out by Hunt, Sauer, and Yorke [23] that there is no canonical choice of a probability measure on bounded subsets of an infinite-dimensional space, which corresponds to the Lebesgue measure in finite dimensions. Therefore, Maier-Paape and Wanner [26] used the following concept of probability. In a small neighborhood of the homogeneous equilibrium there exists a finite-dimensional inertial manifold of the Cahn–Hilliard equation which exponentially attracts all nearby orbits. Thus, if we observe an orbit, we actually observe only its projection onto this manifold. On this manifold, however, we have a canonical probability measure induced by the finite-dimensional Lebesgue measure, and this is used to quantify the probability statement in Theorem 3.6. For more details we refer the reader to [26].

4. Conclusions and open questions. The large size of the radius for which we can explain spinodal decomposition is considerably better than in the results of Maier-Paape and Wanner; their results gave both a starting and ending radius of order ε^n. Our result even gives an end radius which is nondecreasing as $\varepsilon \to 0$. More importantly, our result is qualitatively more illuminating. We show that spinodal decomposition is a result of the fact that most nonlinear solutions are forced into a region of phase space in which the nonlinearity has little effect. These results are supported by numerical simulations; see Sander and Wanner [31]. We end this section with some questions which remain open.

As motivated in the introduction, the $H^2(\Omega)$-norm is the mathematically relevant norm to consider for the Cahn–Hilliard equation. The relationship between this norm and the $L^\infty(\Omega)$-norm is subtle. However, our numerical results in [31] indicate that in one space dimension, solutions starting near equilibrium and measured at some specified $H^2(\Omega)$-end radius have an $L^\infty(\Omega)$-norm proportional to ε^{-2}.

Aside from the slight improvement of the iterative method mentioned in Remark 2.12, our estimate appears to be as good as possible for solutions restricted to cones $\mathcal{K}_{\delta_\varepsilon}$ with respect to the described splitting $X^{1/2} = X_\varepsilon^+ \oplus X_\varepsilon^-$. We believe we can improve these results by adapting (reducing) the dimension of the dominating subspace X_ε^+ as the radius increases. Another potential way to improve on this order is to come up with another appropriate almost linear region, into which most solutions of small radius are driven. However, even without these modifications, the method is both powerful and general. We are optimistic that it can be applied to a variety of other equations.

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