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Pattern formation in a nonlinear model for animal coats

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Abstract

Several models have been proposed for describing the formation of animal coat patterns. We consider reaction–diffusion models due to Murray, which rely on a Turing instability for the pattern selection. In this paper, we describe the early stages of the pattern formation process for large domain sizes. This includes the selection mechanism and the geometry of the patterns generated by the nonlinear system on one-, two-, and three-dimensional base domains. These results are obtained by an adaptation of results explaining the occurrence of spinodal decomposition in materials science as modeled by the Cahn–Hilliard equation. We use techniques of dynamical systems, viewing solutions of the reaction–diffusion model in terms of nonlinear semiflows. Our results are applicable to any parabolic system exhibiting a Turing instability.

MSC: 35K50; 35K55; 35K57; 35B25; 92B05

Keywords: Animal coat pattern; Reaction–diffusion system; Turing instability; Pattern formation

1. Introduction

Understanding and explaining both the formation and the diversity of animal coat patterns has intrigued biologists and mathematicians alike. Several mathematical models have been proposed, relying on different biological and chemical processes. One of the first models is an activator–inhibitor model due to Murray [18–20]. He

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suggests a reaction–diffusion system of the form

$$u_t = \Delta u + \gamma \cdot f(u,v) \quad \text{in } \Omega,$$

$$v_t = d \cdot \Delta v + \gamma \cdot g(u,v) \quad \text{in } \Omega, \tag{1}$$

subject to homogeneous Neumann boundary conditions, where d and γ are positive constants modeling the ratio of diffusion coefficients and the effect of domain size, respectively. As for the specific choice of the nonlinearities, Murray [20] suggests the use of the Thomas system nonlinearities given by

$$f(u,v) = a - u - h(u,v),$$

$$g(u,v) = \alpha(b - v) - h(u,v),$$

$$h(u,v) = \frac{\rho u v}{1 + u + K u^2}, \tag{2}$$

where a, b, α, ρ, and K are positive parameters. With these specific nonlinearities system (1) was originally proposed as a model for chemical reactions involving oxygen and uric acid in the presence of the enzyme uricase. See [26]. Roughly speaking, the geometry of the function u in this model can be interpreted as describing the coat pattern of a specific mammal, with Ω defining the normalized shape of the animal's coat and γ being a measure of its actual size (obtained via rescaling space and time). Similar models have also been used to explain shell patterns [16,17].

For a large range of the above parameters, the reaction–diffusion system (1) has an unstable spatially homogeneous equilibrium $(\bar{u}_0, \bar{v}_0)$. Murray suggests that due to fluctuations during the embryonic stage of a mammal, one has to consider initial conditions (u_0, v_0) close to this homogeneous equilibrium $(\bar{u}_0, \bar{v}_0)$, and study the evolution of the corresponding solution of (1). In this way, randomness enters the otherwise deterministic pattern formation process, which is commonly referred to as the Turing mechanism. See [27]. A good model for this process must have the property that starting at the random initial condition (u_0, v_0), the corresponding solution of (1) is quickly driven away from the homogeneous equilibrium $(\bar{u}_0, \bar{v}_0)$, thereby developing a characteristic pattern. The fine structure of this pattern must depend crucially on the specific initial condition (u_0, v_0). Nevertheless, the key features of the generated patterns must remain the same for different initial conditions close to $(\bar{u}_0, \bar{v}_0)$. Such behavior reflects the fact that different species exhibit different types of animal coats, while the patterns within a species show a high level of individuality.

Numerical simulations of the Thomas system (1) and (2) in Murray [18–20] indicate that the above heuristics are true and that the generated patterns

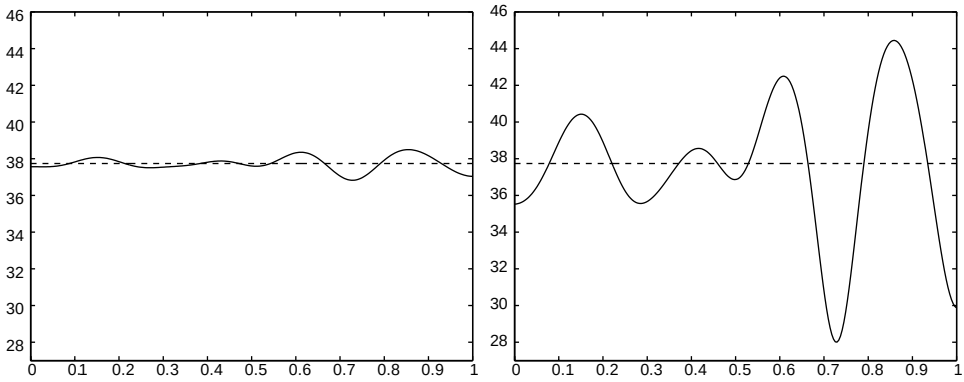

Fig. 1. Simulation of the Thomas system for the one-dimensional case $\Omega = (0,1)$ with $a = 150$, $b = 100$, $\alpha = 1.5$, $\rho = 13$, $K = 0.05$, $d = 500$, and $\gamma = 3500$. The two figures show the u-component at times $t = 0.002$ and 0.003 of a solution originating at a random perturbation near the homogeneous equilibrium. Notice that the qualitative form of u is the same for both times; only the amplitude grows.

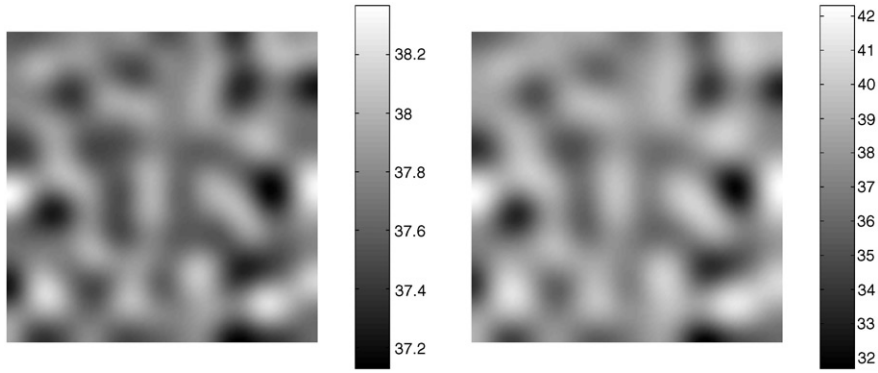

Fig. 2. Simulation of the Thomas system for $\Omega = (0,1)^2$ with the same parameters as in Fig. 1. The two figures show the u-component at times $t = 0.002$ and $t = 0.003$ of a solution originating at a random perturbation near the homogeneous equilibrium. Just as in the previous figure, the qualitative form of u is the same for both times; only the amplitude grows.

qualitatively agree with observed animal coat patterns. See Figs. 1 and 2, which depict simulations for one- and two-dimensional domains Ω.

How can these patterns—such as the complicated two-dimensional ones shown in Fig. 2—be explained mathematically? In [20, Chapters 14 and 15], Murray gives a heuristic explanation for the formation of animal coat patterns by analyzing the linearization of the reaction–diffusion model (1) at the homogeneous equilibrium $(\bar{u}_0, \bar{v}_0)$. He proposes that only the eigenfunction corresponding to the largest eigenvalue of the linearization determines the observed patterns. For one-dimensional domains Ω this description of the observed patterns is qualitatively

correct. Murray admits, however, that this method cannot explain the complicated irregular patterns in two dimensions: "If the dominant eigenfunction is genuinely two-dimensional …, then the full nonlinear spatial pattern is not in general predicted by the linear analysis: It depends on the initial conditions and the nonlinearities in the reaction scheme" [18, p. 173f]. See also [20, p. 396]. In contrast to Murray's conjecture, we show here that the patterns in one, two, and three dimensions *can* be explained by looking at linear behavior—not just corresponding to the largest eigenvalue, but instead corresponding to a whole range of largest eigenvalues.

The above situation is reminiscent of a scenario occurring in the seemingly unrelated field of materials science. We will show that the two situations are truly analogous. After the sudden cooling of a high-temperature binary metal alloy one can observe a phase separation process called spinodal decomposition, which generates patterns very similar to the ones shown in Figs. 1 and 2. Mathematically, spinodal decomposition is modeled by the Cahn–Hilliard equation [3,4]. This is a fourth-order parabolic partial differential equation for an order parameter u describing the local concentration of one of the two metallic components. Similar to the case of animal coat patterns, the first mathematical treatment of spinodal decomposition due to Grant [7] considered only one-dimensional domains Ω. Grant's method employs the same heuristics as described above and he derives precise statements on both the dynamical behavior leading to phase separation and the generated geometries for the nonlinear Cahn–Hilliard equation. Nevertheless, his method cannot explain the complicated patterns which are generated on higher-dimensional domains Ω. This higher-dimensional case was addressed in a series of papers by Maier-Paape and Wanner [14,15], and Sander and Wanner [23,24]. See also the survey paper [12]. These later results can be described as follows. During the early stages of the decomposition process, the nonlinear pattern formation can be fully understood by studying solutions to the linearized system. This is quantified in detail in [14,15] and shows that the observed patterns arise as random superpositions of a finite set of suitable eigenfunctions of the Laplacian on Ω, which form the so-called *dominating subspace*. The subsequent stages of the decomposition process are the subject of [23,24]. There it is shown that even though the solution of the nonlinear Cahn–Hilliard equation is already far from equilibrium, its dynamics are still determined by the linearization at the homogeneous equilibrium. This is due to the fact that the initial decomposition process drives the solution into a region of phase space where linear behavior prevails. This explains both the occurrence and the geometry of the complicated higher-dimensional patterns during spinodal decomposition.

In this paper, we show that the same mechanism which is responsible for spinodal decomposition in the Cahn–Hilliard equation is responsible for the patterns seen in the reaction–diffusion models for the formation of animal coat patterns. The adaptation of the spinodal decomposition results is a powerful tool for understanding reaction–diffusion systems. It allows us in particular to identify the complicated patterns as superpositions of certain eigenfunctions, and to draw conclusions about their geometry. All of these results hold asymptotically for large values of the parameter γ, i.e., for large domain sizes. Moreover, our results are

applicable to arbitrary parabolic systems of form (1) which exhibit a Turing instability.

The remainder of the paper is organized as follows. In Section 2 we state the precise assumptions and our main results. In particular, we recall the conditions for the occurrence of a Turing instability in parabolic systems. In order to simplify the presentation, we do not consider systems of form (1) for large values of γ. Rather, we rescale time and consider the new parameter $\varepsilon = 1/\gamma$. Section 3 contains properties of the linearized equation, in particular of its eigenfunctions and eigenvalues. Section 4 describes the phase space in which the system generates a nonlinear semiflow. The following two sections contain the proofs of our main results. In Section 5 we show that the results of Maier-Paape and Wanner [14,15] can be carried over to the case of parabolic systems. Even though the case of systems of Cahn–Hilliard equations has already been treated in [13], the present situation is more complicated due to the lack of self-adjointness of the linearization. These results describe the early stages of the pattern formation process in domains of arbitrary dimension. The subsequent stages are the subject of Section 6. Here, we employ the methods of Sander and Wanner [23,24]. We conclude with some remarks and open questions in Section 7.

2. Statement of the results

In order to simplify our presentation, we rescale the parabolic system (1) by introducing the new time variable $\tilde{t} = \gamma \cdot t$. If we drop the tilde after rescaling and introduce the new parameter

$$\varepsilon = \frac{1}{\gamma} \tag{3}$$

then (1) is equivalent to the parabolic system

$$\begin{aligned} u_t &= \varepsilon \cdot \Delta u + f(u,v) \quad \text{in } \Omega, \\ v_t &= d \cdot \varepsilon \cdot \Delta v + g(u,v) \quad \text{in } \Omega, \end{aligned} \tag{4}$$

subject to homogeneous Neumann boundary conditions. As mentioned in the introduction, we study (1) for large values of γ, i.e., in view of (3) we are interested in the behavior of (4) for small values of $\varepsilon > 0$. Our basic assumptions on the domain Ω and the nonlinearities f and g are as follows.

Assumption 2.1 (Domain). We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz continuous boundary, where $n \in \{1,2,3\}$. Furthermore, we make a technical assumption on the growth of the eigenfunctions for the negative Laplacian, satisfied for example for all rectangular domains. This is stated precisely in Assumption 3.2.

Assumption 2.2 (Smoothness of the nonlinearity). Let $\sigma \in \mathbb{N}$ be arbitrary. (When applying this to the Thomas system, we choose $\sigma = 1$.) Assume that $f, g : \mathbb{R}^2 \to \mathbb{R}$ are $C^{1+\sigma}$-functions, and that there exists a point $(\bar{u}_0, \bar{v}_0) \in \mathbb{R}^2$ with $f(\bar{u}_0, \bar{v}_0) = g(\bar{u}_0, \bar{v}_0) = 0$. If $\sigma \geqslant 2$ we assume further that all partial derivatives of f and g of order $2, 3, \ldots, \sigma$ at $(\bar{u}_0, \bar{v}_0)$ vanish.

Assumption 2.2 implies that the constant function $(\bar{u}_0, \bar{v}_0)$ is an equilibrium solution of (4) for arbitrary $\varepsilon > 0$. Furthermore, standard results show that under the above assumptions, the parabolic system (4) generates a nonlinear semiflow in a suitable phase space $\mathbb{X}^\alpha$ [9,21]. See Section 4.

Recall that a system of form (4) exhibits a Turing instability, if it has a homogeneous equilibrium solution which is stable in the absence of diffusion, but unstable for $\varepsilon > 0$. This is guaranteed by the following conditions.

Assumption 2.3 (Turing Instability). Let f and g be as in Assumption 2.2, and assume that for some constant $d > 0$ we have

$$f_u + g_v < 0, \tag{5}$$

$$f_u g_v - f_v g_u > 0, \tag{6}$$

$$d f_u + g_v > 0, \tag{7}$$

$$(d f_u + g_v)^2 - 4d(f_u g_v - f_v g_u) > 0 \tag{8}$$

as well as

$$f_u > 0, \tag{9}$$

$$(f_u + g_v)^2 - 4(f_u g_v - f_v g_u) > 0. \tag{10}$$

All partial derivatives in these estimates are evaluated at the point $(\bar{u}_0, \bar{v}_0)$. Notice that due to (5), (7), and (9) we must have $d > 1$.

The first four inequalities in Assumption 2.3 are the standard conditions for the occurrence of a Turing instability. Estimates (5) and (6) guarantee the stability of the homogeneous equilibrium in the absence of diffusion, while (7) and (8) are responsible for the instability of the homogeneous equilibrium. See the lemma below. Inequality (9) can be assumed without loss of generality: Due to (5) and (7) the partial derivatives f_u and g_v must have opposite sign, and therefore (9) can always be achieved by possibly rescaling (4) and exchanging both u and v, as well as f and g. Finally, estimate (10) implies that in the absence of diffusion the eigenvalues

of the linearization are real. It is satisfied in particular for the Thomas system nonlinearities (2).

Lemma 2.4. *Suppose that system* (4) *satisfies Assumptions* 2.1 *and* 2.3. *Then there exists an* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_0$ *the homogeneous equilibrium* $(\bar{u}_0, \bar{v}_0)$ *is unstable*.

This lemma can be proved directly. However, we do not give a direct proof since it follows readily from the results in Section 3. The lemma implies that if we choose an initial condition (u_0, v_0) close to the homogeneous equilibrium $(\bar{u}_0, \bar{v}_0)$, then the corresponding solution of (4) is likely to be driven away from the equilibrium. The following two theorems are the main results of the paper. They describe in more detail how this happens, as well as which solution patterns can be observed during this process.

Our first result describes the early stages of the pattern formation process in the spirit of [14,15]. The proof is in Section 5.3. See Theorem 5.7 for a more detailed version. The result shows that up to a certain distance from the equilibrium, most solutions of (4) originating near the homogeneous equilibrium exhibit patterns similar to the ones in a *dominating subspace* $\mathscr{Y}_\varepsilon^+$. Distances are measured with respect to the special norm $\|\cdot\|_*$ on $\mathbb{X}^\alpha$ described in Section 4, but closeness with respect to this norm implies closeness with respect to the L^∞-norm. See Lemma 4.4. Functions in the dominating subspace $\mathscr{Y}_\varepsilon^+$ exhibit patterns similar to the ones depicted in Figs. 1 and 2. See Section 5 for more details.

Theorem 2.5 (Early Pattern Formation). *Assume that system* (4) *satisfies Assumptions* 2.1–2.3. *Choose any constant* α *with* $\dim \Omega/4 < \alpha < 1$, *and let* $0 < d_0 \ll 1$ *be arbitrary, but fixed. Then there exists an* $\varepsilon_0 > 0$, *such that for every* $0 < \varepsilon \leqslant \varepsilon_0$ *there exists a finite-dimensional subspace* $\mathscr{Y}_\varepsilon^+$, *as well as radii* $0 < r_\varepsilon < R_\varepsilon$ *such that the following is true. In the limit* $\varepsilon \to 0$ *we have*

$$\dim \mathscr{Y}_\varepsilon^+ \sim \varepsilon^{-\dim \Omega/2}, \quad r_\varepsilon \sim \varepsilon^{(2\alpha+\dim \Omega)/(2\sigma)}, \quad \text{and} \quad R_\varepsilon \sim \varepsilon^{(2\alpha+\dim \Omega)/(2\sigma)}.$$

Furthermore, for most initial conditions $(u_0, v_0) \in \mathbb{X}^\alpha$ *satisfying*

$$\|(u_0, v_0) - (\bar{u}_0, \bar{v}_0)\|_* < r_\varepsilon,$$

the corresponding solution (u, v) *of* (4) *exits a ball around the homogeneous equilibrium* $(\bar{u}_0, \bar{v}_0)$ *of radius* R_ε, *and upon exiting the ball the distance of* (u, v) *to the subspace* $\mathscr{Y}_\varepsilon^+$ *is at most* $d_0 \cdot R_\varepsilon$. *In the above estimate, the norm* $\|\cdot\|_*$ *denotes the norm on* $\mathbb{X}^\alpha$ *which is introduced in Definition* 4.1 *and Proposition* 4.3.

Remark 2.6. Let us add a few comments on the precise meaning of "most initial conditions" in the above theorem. It was pointed out by Hunt et al. [10] that there is no canonical choice of probability measure on bounded subsets of infinite-dimensional spaces, which corresponds to the Lebesgue measure in finite dimensions.

Therefore, Maier-Paape and Wanner [14] used the following concept of probability. In a small neighborhood of the homogeneous equilibrium $(\bar{u}_0, \bar{v}_0)$ there exists a finite-dimensional inertial manifold for (4) which exponentially attracts all nearby orbits. Thus, if we observe an orbit, we actually observe only its projection onto this manifold. On this manifold however, a canonical probability measure is induced by the finite-dimensional Lebesgue measure, and this is used to quantify the probability statement in Theorem 2.5. For more details we refer the reader to [14].

In one-, two-, and three-dimensional domains, the results in Theorem 2.5 can be extended further from equilibrium by adapting the techniques in [24]. The following theorem shows that up to large distances from the homogeneous equilibrium, the behavior of the nonlinear system (4) is basically linear. Thus, the linear patterns observed persist and only grow in amplitude. The proof of this result is contained in Section 6.3. The result is in agreement with our simulations in Figs. 1 and 2.

Theorem 2.7 (Almost Linear Behavior). *Consider the reaction–diffusion system* (4), *assume that Assumptions* 2.1–2.3 *are satisfied, and let* $\rho > 0$ *be arbitrarily small, but fixed. Let* (u_0, v_0) *denote an initial condition close to the homogeneous equilibrium* $(\bar{u}_0, \bar{v}_0)$, *which is sufficiently close to the dominating subspace* $\mathscr{Y}_\varepsilon^+$. *Finally, let* (u, v) *and* $(u_{\text{lin}}, v_{\text{lin}})$ *be the solutions to* (4) *and its affine approximation at* $(\bar{u}_0, \bar{v}_0)$, *respectively, starting at* (u_0, v_0). *Then the solution* (u, v) *remains close to* $(u_{\text{lin}}, v_{\text{lin}})$ *until the distance from* (u, v) *to the homogeneous equilibrium exceeds a certain threshold. More specifically, as long as*

$$\|(u(t), v(t)) - (\bar{u}_0, \bar{v}_0)\|_* \leq C \cdot \varepsilon^{-(\alpha - \dim \Omega/4) + \alpha/\sigma + \rho} \cdot \|(u_0, v_0) - (\bar{u}_0, \bar{v}_0)\|_*^\rho$$

we have

$$\frac{\|(u(t), v(t)) - (u_{\text{lin}}(t), v_{\text{lin}}(t))\|_*}{\|(u_{\text{lin}}(t), v_{\text{lin}}(t)) - (\bar{u}_0, \bar{v}_0)\|_*} \leqslant C \cdot \varepsilon^{\alpha - \dim \Omega/4}.$$

As shown in [24] it is possible to combine Theorems 2.5 and 2.7 in order to show that at the end of the first pattern formation stage, most solutions of (4) originating near the homogeneous equilibrium satisfy the assumptions of Theorem 2.7, i.e., they stay close to the corresponding solutions of the linearized equation up to large distances from $(\bar{u}_0, \bar{v}_0)$. For more details we refer the reader to [24, Section 3.4].

3. Properties of the linearization

We start by describing the behavior of solutions of the linearization of (4) at the homogeneous equilibrium $(\bar{u}_0, \bar{v}_0)$. This linearization is given by the

parabolic system

$$u_t = \varepsilon \cdot \Delta u + f_u(\bar{u}_0, \bar{v}_0)u + f_v(\bar{u}_0, \bar{v}_0)v \quad \text{in } \Omega,$$

$$v_t = d \cdot \varepsilon \cdot \Delta v + g_u(\bar{u}_0, \bar{v}_0)u + g_v(\bar{u}_0, \bar{v}_0)v \quad \text{in } \Omega, \tag{11}$$

subject to homogeneous Neumann boundary conditions on $\partial\Omega$. In order to abbreviate our notation, let

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} f_u(\bar{u}_0, \bar{v}_0) & f_v(\bar{u}_0, \bar{v}_0) \\ g_u(\bar{u}_0, \bar{v}_0) & g_v(\bar{u}_0, \bar{v}_0) \end{pmatrix}.$$

Then using the abbreviation $U = (u, v)$, the linearized system (11) can be expressed more concisely as

$$U_t = \varepsilon \cdot D\Delta U + BU.$$

Notice that due to (5) and (6) we have both $\operatorname{trace}(B) < 0$ and $\det(B) > 0$. Thus the equilibrium $(\bar{u}_0, \bar{v}_0)$ is stable in the absence of diffusion (i.e., for $\varepsilon = 0$). Eq. (10) implies that the eigenvalues of B are real and distinct. In discussing the spectrum of (11) for $\varepsilon > 0$, we use the following notation for the spectrum of the negative Laplacian acting on real-valued functions.

Definition 3.1 (Eigenvalues of the Negative Laplacian). Let $\Omega \subset \mathbb{R}^n$ be a domain as in Assumption 2.1, and consider the self-adjoint linear operator $-\Delta : L^2(\Omega) \to L^2(\Omega)$ subject to homogeneous Neumann boundary conditions. We denote by $0 = \kappa_1 < \kappa_2 < \cdots \to \infty$ the ordered sequence of eigenvalues of $-\Delta$, and the corresponding pairwise orthogonal $L^2(\Omega)$-normalized real-valued eigenfunctions by ϕ_k, for $k \in \mathbb{N}$.

Under Assumption 2.1, the asymptotic growth of the eigenvalues κ_k is known exactly. According to Courant and Hilbert [5, p. 442] or Edmunds and Evans [6] one has

$$\kappa_k \sim k^{2/\dim \Omega} \quad \text{as } k \to \infty. \tag{12}$$

We now formulate an additional technical assumption on the domain Ω. This assumption will only be needed in Section 6, specifically in the proof of Proposition 6.2.

Assumption 3.2. Assume that the domain $\Omega \subset R^n$ is such that the $L^\infty(\Omega)$-norm of the eigenfunctions ϕ_k is uniformly bounded.

Remark 3.3. It can easily be verified that the above assumption is true for all rectangular domains. Unfortunately, it fails to hold for arbitrary domains, one counterexample being a disc in $\mathbb{R}^2$. It has been conjectured in Aurich et al. [2] that for

a large class of domains the $L^\infty(\Omega)$-norm of the eigenfunctions ϕ_k grows asymptotically like $\log \kappa_k$ for $k \to \infty$. For domains exhibiting such logarithmic growth our main result Theorem 6.3 continues to hold.

We now describe the spectrum of the right-hand side of (11). To this end, we begin by considering the matrix $M(s) = B - s \cdot D$ for $s \geqslant 0$. As we later show, for $s = \varepsilon\kappa_k$, the eigenvalues of this matrix are in the spectrum of the right-hand side of the linearized system (11). In fact, Proposition 3.7 shows that this is the entire spectrum.

The characteristic polynomial of the matrix $M(s)$ above is given by

$$\det((B - s \cdot D) - \lambda \cdot I) = \lambda^2 - c_1(s) \cdot \lambda + c_0(s), \tag{13}$$

where

$$c_1(s) = (f_u + g_v) - s \cdot (1 + d), \tag{14}$$

$$c_0(s) = (f_u g_v - g_u f_v) - (d f_u + g_v) \cdot s + d \cdot s^2. \tag{15}$$

As before, all the derivatives in (14) and (15) are evaluated at the point $(\bar{u}_0, \bar{v}_0)$. The following two lemmas contain results concerning the eigenvalues and eigenvectors of the matrix $B - s \cdot D$ for $s \geqslant 0$.

Lemma 3.4. *Suppose that Assumption* 2.3 *is satisfied. Then for arbitrary* $s \geqslant 0$ *the characteristic polynomial* (13) *has two distinct real roots* $\lambda^-(s) < \lambda^+(s)$. *The function* $\lambda^-(\cdot)$ *is strictly decreasing with* $\lambda^-(s) < 0$ *for all* $s \geqslant 0$. *Moreover,* $\lambda^+(\cdot)$ *satisfies* $\lambda^+(0) < 0$. *It has a unique maximal value* $\lambda^+_{\max} > 0$ *which is attained at some* $s_{\max} > 0$. *It also has two zeros* $s_\ell < s_r$.

The asymptotic behavior of the functions $\lambda^\pm$ *is given by* $\lim_{s\to\infty}(\lambda^+(s)/s) = -1$ *and* $\lim_{s\to\infty}(\lambda^-(s)/s) = -d$. *See also Fig.* 3.

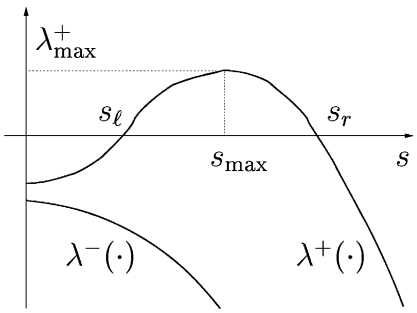

Fig. 3. Behavior of the eigenvalues $\lambda^\pm(s)$ of the matrix $B - s \cdot D$.

Proof. A straightforward calculation shows that (5), (7), and (10) furnish the strict inequality $c_1(s)^2 - 4c_0(s) > 0$ for every $s \geqslant 0$. Thus, (13) has two distinct real roots given by $\lambda^{\pm}(s) = (c_1(s) \pm (c_1(s)^2 - 4c_0(s))^{1/2})/2$. According to (5) we further have $c_1(s) < 0$ for all $s \geqslant 0$, and therefore $\lambda^-(s) < 0$. On the other hand, it can easily be verified that $\lambda^+(s) > 0$ is equivalent to $c_0(s) < 0$. Because of (6)–(8) this last inequality is equivalent to $s_\ell < s < s_r$, where

$$s_{\ell/r} = \frac{1}{2d} \cdot ((df_u + g_v) \mp \sqrt{(df_u + g_v)^2 - 4d(f_u g_v - f_v g_u)}).$$

Finally, a straightforward calculation shows the asymptotic limits for $\lambda^{\pm}(s)/s$. This completes the proof of the lemma. □

For a fixed s, the following lemma gives estimates on the angle between the eigenvectors of $M(s)$. The operator given by the right-hand side of (11) is not self-adjoint. The following lemma will be used later to show that it is close enough to self-adjoint for us to be able to use standard estimates.

Lemma 3.5. *Suppose that Assumption* 2.3 *holds. For* $s \geqslant 0$ *let* $V^{\pm}(s) \in \mathbb{R}^2$ *denote the normalized eigenvectors of the matrix* $B - s \cdot D$ *corresponding to the eigenvalues* $\lambda^{\pm}(s)$, *whose existence is due to Lemma* 3.4. *Then the angle between* $V^-(s)$ *and* $V^+(s)$ *is strictly bounded away from both* 0 *and* π *for all* $s \geqslant 0$. *Moreover, as s approaches* ∞ *the angle approaches a right angle.*

Proof. It can easily be shown that the eigenvectors $V^{\pm}(s)$ can be chosen continuously with respect to s. Furthermore, since the associated eigenvalues $\lambda^{\pm}(s)$ are distinct for all $s \geqslant 0$, we know that the eigenvectors $V^{\pm}(s)$ are linearly independent for all $s \geqslant 0$. We only need to worry about the limit. Note that for a nonzero constant c, the eigenvectors of any matrix A and the scaled matrix cA are the same, even though their eigenvalues differ. Thus, the eigenvectors of $B - s \cdot D$ are the same as the eigenvectors of the matrix $B/s - D$. As $s \to \infty$, the latter eigenvectors approach the eigenvectors of $-D$. Since D is a diagonal matrix, its eigenvectors are the standard unit vectors in $\mathbb{R}^2$, thus orthogonal. □

Now we have gathered everything to describe the spectrum of the right-hand side of (11).

Definition 3.6 (The Linear Operator A_ε). Let $\mathbb{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$, and for arbitrary $s > 0$ define $\mathbb{H}^s(\Omega) = H^s(\Omega) \times H^s(\Omega)$, where $H^s(\Omega)$ denotes the standard fractional Sobolev space for real-valued functions. Define

$$\mathbb{X} = \mathbb{L}^2(\Omega), \tag{16}$$

and let $A_\varepsilon : \mathbb{X} \to \mathbb{X}$ be the linear operator given by the right-hand side of (11) subject to homogeneous Neumann boundary conditions. In other words, we

consider A_ε on the domain $D(A_\varepsilon) = \mathbb{H}^2_N(\Omega)$, where $\mathbb{H}^2_N(\Omega)$ denotes the subspace of $\mathbb{H}^2(\Omega)$ which consists of all functions satisfying homogeneous Neumann boundary conditions on $\partial\Omega$.

Proposition 3.7 (Spectrum of A_ε). *Assume that Assumptions* 2.1 *and* 2.3 *are satisfied. Then $-A_\varepsilon$ is a sectorial operator, and the spectrum of A_ε consists of the eigenvalues $\lambda^\pm_{k,\varepsilon} = \lambda^\pm(\varepsilon\cdot\kappa_k)$ for $k\in\mathbb{N}$, where $\lambda^\pm$ is given as in the above Lemma* 3.4. *The eigenfunctions corresponding to $\lambda^\pm_{k,\varepsilon}$ are given by $\psi^\pm_{k,\varepsilon} = \phi_k\cdot V^\pm_{k,\varepsilon}$, where $V^\pm_{k,\varepsilon} = V^\pm(\varepsilon\cdot\kappa_k)$ and $V^\pm(\cdot)$ was defined in Lemma* 3.5. *These eigenfunctions form a complete set in $\mathbb{X}$.*

Proof. Standard results imply that the operator $-A_\varepsilon$ is sectorial and has a compact resolvent. Thus, the spectrum of A_ε consists completely of eigenvalues with finite multiplicities. Due to Lemmas 3.4 and 3.5 the functions $\psi^\pm_{k,\varepsilon}$ are eigenfunctions of A_ε with corresponding eigenvalues $\lambda^\pm_{k,\varepsilon}$. Moreover, since the vectors $V^\pm(\varepsilon\cdot\kappa_k)\in\mathbb{R}^2$ are linearly independent and the functions ϕ_k, $k\in\mathbb{N}$, form a complete set in $L^2(\Omega)$, the eigenfunctions $\psi^\pm_{k,\varepsilon}$ form a complete set in $\mathbb{L}^2(\Omega)$. □

4. Phase space

In this section, we show that there is a phase space on which the parabolic system in Eq. (4) generates a nonlinear semiflow.

Proposition 3.7 above shows that the linearization A_ε generates an analytic semigroup on the space $\mathbb{X} = \mathbb{L}^2(\Omega)$. For every choice of $\alpha\in(0,1)$ the fractional power space $\mathbb{X}^{\alpha,\varepsilon} = D((aI - A_\varepsilon)^\alpha)\subset\mathbb{X}$ (equipped with the graph norm) is defined for any constant $a > \lambda^+_{\max}$, where $\lambda^+_{\max}$ is as in Lemma 3.4. Spaces of this type play an important role in the discussion of nonlinear equations. See for example Henry [9] and Pazy [21].

At first glance it seems that the space $\mathbb{X}^{\alpha,\varepsilon}$ depends both on α and on ε. While the dependence on α is unavoidable (even desirable), the dependence on ε would be an obstacle to comparing our results for different values of the domain size $\gamma = 1/\varepsilon$. Fortunately, Proposition 4.3 implies that in fact the above fractional power space is ε-independent. First we need some auxiliary considerations.

Definition 4.1 (The space $\mathbb{H}^s_*(\Omega)$). Assume that Assumptions 2.1 and 2.3 are satisfied, and consider the spaces $\mathbb{H}^s(\Omega)\subset\mathbb{L}^2(\Omega)$ defined in Definition 3.6.

For $s\in(0,2)$, let $H^s_*(\Omega)$ denote the closure of the span of the set $\{\phi_k\colon k\in\mathbb{N}\}$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$ of the fractional Sobolev space $H^s(\Omega)$, where the functions ϕ_k are the eigenfunctions of Definition 3.1. Let $\mathbb{H}^s_*(\Omega) = H^s_*(\Omega)\times H^s_*(\Omega)$, equipped with the norm $\|\cdot\|_{\mathbb{H}^s(\Omega)}$ defined by $\|(u,v)\|^2_{\mathbb{H}^s(\Omega)} = \|u\|^2_{H^s(\Omega)} + \|v\|^2_{H^s(\Omega)}$. According to Proposition 3.7 every function $U\in\mathbb{L}^2(\Omega)$ can be written uniquely as a

series of the form

$$U = \sum_{k=1}^{\infty} (\alpha_k^+ \cdot V_{k,\varepsilon}^+ + \alpha_k^- \cdot V_{k,\varepsilon}^-) \cdot \phi_k. \tag{17}$$

When the following is finite, define $||\cdot||_*$ by

$$||U||_*^2 = \sum_{k=1}^{\infty} (1+\kappa_k)^s \cdot ((\alpha_k^+)^2 + (\alpha_k^-)^2). \tag{18}$$

The following lemma describes where $||\cdot||_*$ makes sense as a norm.

Lemma 4.2 (Characterization of $\mathbb{H}_*^s(\Omega)$). *Assume that Assumptions* 2.1 *and* 2.3 *are satisfied. Then* $U \in \mathbb{L}^2(\Omega)$ *of the form in* (17) *is contained in the space* $\mathbb{H}_*^s(\Omega)$ *if and only if* $||U||_* < \infty$. *Furthermore, the* $||\cdot||_*$ *norm is equivalent to the standard norm* $||\cdot||_{\mathbb{H}^s(\Omega)}$, *i.e., there exist* ε*-independent positive constants* C_1 *and* C_2 *such that*

$$C_1 \cdot ||U||_{\mathbb{H}^s(\Omega)} \leqslant ||U||_* \leqslant C_2 \cdot ||U||_{\mathbb{H}^s(\Omega)} \quad \text{for all } U \in \mathbb{H}_*^s(\Omega).$$

Proof. Let $w \in H_*^s(\Omega)$ be scalar-valued. If we write $w = \sum_{k=1}^{\infty} \alpha_k \phi_k$ as a Fourier series in $L^2(\Omega)$, then standard results on fractional Sobolev spaces imply that the definition

$$||w||_{H^s(\Omega)}^2 = \sum_{k=1}^{\infty} (1+\kappa_k)^s \cdot \alpha_k^2$$

furnishes a norm on $H_*^s(\Omega)$, which is equivalent to the usual one. See Lions and Magenes [11] and Temam [25]. For convenience, we use only this norm in the following. We now show the equivalence of norms. Consider $U \in \mathbb{L}^2(\Omega)$ as in (17). Then the above discussion and the definition of the norm on $\mathbb{H}^s(\Omega)$ imply

$$||U||_{\mathbb{H}^s(\Omega)}^2 = \sum_{k=1}^{\infty} (1+\kappa_k)^s \cdot ||\alpha_k^+ \cdot V_{k,\varepsilon}^+ + \alpha_k^- \cdot V_{k,\varepsilon}^-||_{\mathbb{R}^2}^2.$$

Moreover, we have

$$||\alpha_k^+ \cdot V_{k,\varepsilon}^+ + \alpha_k^- \cdot V_{k,\varepsilon}^-||_{\mathbb{R}^2}^2 = (\alpha_k^+)^2 + (\alpha_k^-)^2 + 2\alpha_k^+ \alpha_k^- \cdot (V_{k,\varepsilon}^+, V_{k,\varepsilon}^-)_{\mathbb{R}^2}$$

for arbitrary $k \in \mathbb{N}$ and $\varepsilon > 0$. Due to the fact that the angle between $V_{k,\varepsilon}^+$ and $V_{k,\varepsilon}^-$ is strictly bounded away from both 0 and π for all $k \in \mathbb{N}$ and $\varepsilon > 0$, which is a

consequence of Lemma 3.5 and Proposition 3.7, the equivalence of norms follows from the Cauchy–Schwarz inequality. □

The above characterization of the space $\mathbb{H}^s_*(\Omega)$ allows us finally to describe the fractional power spaces $\mathbb{X}^{\alpha,\varepsilon}$ of A_ε.

Proposition 4.3 (Properties of $\mathbb{X}^{\alpha,\varepsilon}$). *Assume that both Assumption* 2.1 *and* 2.3 *are satisfied, and consider the linear operator* A_ε *from Proposition* 3.7. *Choose a constant* $a > \lambda^+_{\max}$, *with* $\lambda^+_{\max}$ *as in Lemma* 3.4, *and for some* $\alpha \in (0,1)$ *consider the fractional power space* $\mathbb{X}^{\alpha,\varepsilon} = D((aI - A_\varepsilon)^\alpha)$. *Let* $||\cdot||_{\alpha,\varepsilon}$ *denote the standard norm on* $\mathbb{X}^{\alpha,\varepsilon}$ *given by* $||U||_{\alpha,\varepsilon} = ||(aI - A_\varepsilon)^\alpha U||_{\mathbb{L}^2(\Omega)}$.

Then for every $0 < \varepsilon \leqslant 1$ *we have* $\mathbb{X}^{\alpha,\varepsilon} = \mathbb{H}^{2\alpha}_*(\Omega)$, *and the norm* $||\cdot||_{\alpha,\varepsilon}$ *is equivalent to the norm* $||\cdot||_*$ *introduced in Lemma* 4.2 (*with* $s = 2\alpha$). *More precisely, there exist* ε*-independent constants* C_1 *and* C_2 *such that*

$$C_1 \cdot \varepsilon^\alpha \cdot ||U||_* \leqslant ||U||_{\alpha,\varepsilon} \leqslant C_2 \cdot ||U||_* \quad \text{for all } U \in \mathbb{X}^{\alpha,\varepsilon} = \mathbb{H}^{2\alpha}_*(\Omega).$$

Proof. For any function $U \in \mathbb{L}^2(\Omega)$ as in (17) we define

$$T_\varepsilon U = \sum_{k=1}^{\infty} (\alpha_k^+ \cdot e_1 + \alpha_k^- \cdot e_2) \cdot \phi_k, \tag{19}$$

where e_1 and e_2 denote the standard unit vectors in $\mathbb{R}^2$. Using Lemma 3.5 one can show that the mapping $T_\varepsilon : \mathbb{L}^2(\Omega) \to \mathbb{L}^2(\Omega)$ is a bounded and invertible linear operator, and that

$$c_1 \cdot ||U||_{\mathbb{L}^2(\Omega)} \leqslant ||T_\varepsilon U||_{\mathbb{L}^2(\Omega)} \leqslant c_2 \cdot ||U||_{\mathbb{L}^2(\Omega)} \quad \text{for all } U \in \mathbb{L}^2(\Omega) \tag{20}$$

with ε-independent positive constants c_1 and c_2. This crucial observation is again due to the fact that the angle between $V^+(s)$ and $V^-(s)$ is strictly bounded away from both 0 and π for all $s \geqslant 0$—and it follows as in the proof of Lemma 4.2.

Now consider the unbounded operator $B_\varepsilon = T_\varepsilon A_\varepsilon T_\varepsilon^{-1}$. Using a result due to Riesz and Sz.-Nagy [22, Section 120, Lemma, p. 314] it is not too hard to show that the operator B_ε is self-adjoint. Furthermore, Theorem 4.6.7 in Amann [1, Chapter III] can be used to obtain a characterization of the domain of the fractional power $(aI - B_\varepsilon)^\alpha$ in terms of Fourier coefficients. Together with the identity $(aI - A_\varepsilon)^\alpha = T_\varepsilon^{-1}(aI - B_\varepsilon)^\alpha T_\varepsilon$ this finally implies that a function $U \in \mathbb{L}^2(\Omega)$ with the series representation (17) is contained in the fractional power space $\mathbb{X}^{\alpha,\varepsilon} = D((aI - A_\varepsilon)^\alpha)$ if and only if $T_\varepsilon U$ is contained in $D((aI - B_\varepsilon)^\alpha)$; this in turn is equivalent to

$$||U||^2_\# = \sum_{k=1}^{\infty} ((a - \lambda^+_{k,\varepsilon})^{2\alpha} \cdot (\alpha_k^+)^2 + (a - \lambda^-_{k,\varepsilon})^{2\alpha} \cdot (\alpha_k^-)^2) < \infty. \tag{21}$$

Furthermore, the norm $||U||_{\alpha,\varepsilon}$ for a function U as in (17) is given by

$$\begin{aligned} ||U||_{\alpha,\varepsilon} &= \left|\left| \sum_{k=1}^{\infty} ((a-\lambda_{k,\varepsilon}^{+})^{\alpha} \cdot \alpha_k^{+} \cdot V_{k,\varepsilon}^{+} + (a-\lambda_{k,\varepsilon}^{-})^{\alpha} \cdot \alpha_k^{-} \cdot V_{k,\varepsilon}^{-}) \cdot \phi_k \right|\right|_{\mathbb{L}^2(\Omega)} \\ &= \left|\left| T_{\varepsilon}^{-1} \sum_{k=1}^{\infty} ((a-\lambda_{k,\varepsilon}^{+})^{\alpha} \cdot \alpha_k^{+} \cdot e_1 + (a-\lambda_{k,\varepsilon}^{-})^{\alpha} \cdot \alpha_k^{-} \cdot e_2) \cdot \phi_k \right|\right|_{\mathbb{L}^2(\Omega)}, \end{aligned}$$

and this implies with (20) and (21) that

$$c_1 \cdot ||U||_{\alpha,\varepsilon} \leqslant ||U||_{\#} \leqslant c_2 \cdot ||U||_{\alpha,\varepsilon} \quad \text{for all } U \in \mathbb{X}^{\alpha,\varepsilon}. \tag{22}$$

So far we have shown that a function $U \in \mathbb{L}^2(\Omega)$ with representation (17) is in the fractional power space $\mathbb{X}^{\alpha,\varepsilon}$ if and only if (21) is satisfied. Moreover, we know that $||\cdot||_{\#}$ and $||\cdot||_{\alpha,\varepsilon}$ are equivalent norms on $\mathbb{X}^{\alpha,\varepsilon}$ via (22), with ε-independent positive constants c_1 and c_2.

In order to complete the proof we have to relate these results to the space $\mathbb{H}_*^{2\alpha}(\Omega)$ from Lemma 4.2 and its norm $||\cdot||_*$. To this end, recall the explicit representation of the eigenvalues $\lambda_{k,\varepsilon}^{\pm} = \lambda^{\pm}(\varepsilon \cdot \kappa_k)$ due to Lemma 3.4 and Proposition 3.7, which is given in terms of (13)–(15). Since the function $c_0(s)$ is quadratic in s, and $c_1(s)$ is linear in s, it follows that for some positive constants c_3 and c_4 we have

$$0 < c_3 \leqslant \frac{a-\lambda^{\pm}(s)}{1+s} \leqslant c_4 \quad \text{for all } s \geqslant 0.$$

Using $\varepsilon \leqslant 1$ this immediately implies $c_3 \cdot \varepsilon \cdot (1+\kappa_k) \leqslant a - \lambda^{\pm}(\varepsilon \cdot \kappa_k) \leqslant c_4 \cdot (1+\kappa_k)$. Together with (18) and (21) this completes the proof of the proposition. □

Due to the above proposition, the fractional power space $\mathbb{X}^{\alpha,\varepsilon}$ is algebraically and topologically independent of ε, even though the norms $||\cdot||_{\alpha,\varepsilon}$ do depend on ε. Therefore, we omit the superscript ε from now on and write simply $\mathbb{X}^{\alpha} = \mathbb{H}_*^{2\alpha}(\Omega)$. Furthermore, we always consider this space equipped with the norm $||\cdot||_*$ defined in Definition 4.1 (with $s = 2\alpha$). The next result provides more information on $\mathbb{X}^{\alpha}$ in terms of classical function spaces.

Lemma 4.4 (Embeddings of $\mathbb{X}^{\alpha}$). *Assume that Assumptions* 2.1 *and* 2.3 *are satisfied, and consider the fractional power space* $\mathbb{X}^{\alpha}$ *as in Proposition* 4.3, *equipped with the norm* $||\cdot||_*$. *Then for every* α *satisfying* $\dim \Omega/4 < \alpha < 1$ *we have*

$$\mathbb{H}_N^2(\Omega) \subset \mathbb{X}^{\alpha} \subset C(\bar{\Omega}) \quad (\textit{continuous embeddings}),$$

where $\mathbb{H}^2_N(\Omega)$ *was defined in Definition* 3.6, *and there exist* ε*-independent constants* C_1 *and* C_2 *such that*

$$C_1 \cdot ||U||_{C(\bar{\Omega})} \leqslant ||U||_* \leqslant C_2 \cdot ||U||_{\mathbb{H}^2(\Omega)} \quad \text{for all } U \in \mathbb{X}^\alpha.$$

Proof. According to $\mathbb{X}^\alpha = \mathbb{H}^{2\alpha}_*(\Omega) \subset \mathbb{H}^{2\alpha}(\Omega)$, this result follows immediately from Sobolev's embedding theorem for fractional Sobolev spaces. See [11,25]. □

5. The early pattern formation stages

In this section, we describe the behavior of solutions of the nonlinear system (4) starting near the homogeneous equilibrium $(\bar{u}_0, \bar{v}_0)$. Due to Lemma 2.4 we know that they are likely to be driven away, and due to the numerical simulations presented in the introduction we expect them to develop complicated geometric patterns. Based on the results by Maier-Paape and Wanner [15] we prove Theorem 2.5, which was formulated in Section 2. In order to apply the theory of [15] we have to formulate (4) as an abstract evolution equation of the form

$$U_t = A_\varepsilon U + F(U) \tag{23}$$

on a suitable function space. To this end, consider the Hilbert space $\mathbb{X}$ defined in (16), and let $A_\varepsilon : \mathbb{X} \to \mathbb{X}$ be as in Definition 3.6. Then the results of the last two sections imply that $-A_\varepsilon$ is a sectorial operator, and that the corresponding fractional power space $\mathbb{X}^\alpha$ is given by the space $\mathbb{H}^{2\alpha}_*(\Omega)$ defined in Definition 4.1. As pointed out after Proposition 4.3, we equip this space with the norm $||\cdot||_*$ defined in (8). As for the nonlinearity F, we define the function $h : \mathbb{R}^2 \to \mathbb{R}^2$ to be the nonlinear part of (f, g). That is

$$\hat{h}(u, v) = (f(u, v), g(u, v))$$

and

$$h(u, v) = \hat{h}(u, v) - \hat{h}_u(\bar{u}_0, \bar{v}_0) \cdot (u - \bar{u}_0) - \hat{h}_v(\bar{u}_0, \bar{v}_0) \cdot (v - \bar{v}_0). \tag{24}$$

If we now set $F(U) = h(u, v)$ for $U = (u, v) \in \mathbb{X}^\alpha$, then formally Eq. (4) is of form (23) above.

In order to apply the results of [15] we have to verify hypotheses (H1) through (H3) in [15]. In summary, we need to verify the following:

(H1) The operator $-A_\varepsilon$ is a sectorial operator on $\mathbb{X}$.

(H2) There exists a decomposition $\mathbb{X} = \mathbb{X}^{--} \oplus \mathbb{X}^{-} \oplus \mathbb{X}^{+} \oplus \mathbb{X}^{++}$, such that all of these subspaces are finite-dimensional except $\mathbb{X}^{--}$, and such that the linear semigroup corresponding to $U_t = A_\varepsilon U$ satisfies several dichotomy estimates.

(H3) The nonlinearity $F : \mathbb{X}^\alpha \to \mathbb{X}$ is continuously differentiable, and satisfies both $F(\bar{u}_0, \bar{v}_0) = 0$ and $DF(\bar{u}_0, \bar{v}_0) = 0$.

While (H1) has already been addressed in Proposition 3.7, we still have to verify (H2) and (H3). This is done in the following two subsections. In Section 5.3, we prove Theorem 2.5.

5.1. Decomposition of the phase space

We begin with the following lemma, which establishes the existence of certain spectral gaps in the spectrum of A_ε. Since the results of [15] rely on the theory of invariant manifolds and foliations, the size of these spectral gaps has profound implications for the size of the neighborhood on which the linear behavior dominates. Thus, the gaps have to be carefully analyzed.

Lemma 5.1 (Spectral Gaps for A_ε). *Suppose that Assumptions* 2.1 *and* 2.3 *are satisfied. Choose two arbitrary constants* $c_* < c^* \leqslant \lambda^+_{\max}$, *where* $\lambda^+_{\max}$ *is as in Lemma* 3.4. *Then there exist positive constants* ε_0 *and* d_0 *such that for arbitrary* $0 < \varepsilon \leqslant \varepsilon_0$ *the following is true. The linear operator* A_ε *has eigenvalues* $\lambda_*(\varepsilon)$ *and* $\lambda^*(\varepsilon)$ *which satisfy* $c_* \leqslant \lambda_*(\varepsilon) < \lambda^*(\varepsilon) \leqslant c^*$ *and such that the estimate* $\lambda^*(\varepsilon) - \lambda_*(\varepsilon) \geqslant d_0 \cdot \varepsilon^{\dim \Omega/2}$ *holds. Moreover, the interval* $(\lambda_*(\varepsilon), \lambda^*(\varepsilon))$ *is contained in the resolvent set of* A_ε.

Proof. Let $c_1 < c_2 \leqslant \lambda^+_{\max}$ be arbitrary. Then due to Lemma 3.4 and Proposition 3.7 there exist two nonempty compact subsets $I^\pm \subset \mathbb{R}^+_0$ such that $\lambda^\pm_{k,\varepsilon} \in [c_1, c_2]$ if and only if $\kappa_k \in \varepsilon^{-1} \cdot I^\pm$. The subset I^- consists of one compact interval, and I^+ is the union of at most two compact intervals. In combination with the asymptotic distribution of the eigenvalues κ_k given in (12) this implies that as $\varepsilon \to 0$, the number of eigenvalues of A_ε in the interval $[c_1, c_2]$ is of the order $\varepsilon^{-\dim \Omega/2}$.

Choose and fix two constants c_{**} and c^{**} such that $c_* < c_{**} < c^{**} < c^*$. As above we can choose $\varepsilon_0 > 0$ small enough so that for all $0 < \varepsilon \leqslant \varepsilon_0$ both the interval $[c^{**}, c^*]$ and the interval $[c_*, c_{**}]$ contains at least one eigenvalue of A_ε. Let $\lambda^{**}(\varepsilon)$ denote the smallest of these eigenvalues in the first interval, and $\lambda_{**}(\varepsilon)$ the largest one in the second interval. Furthermore, there exists a constant $C > 0$ such that the number of eigenvalues of A_ε in the interval (c_{**}, c^{**}) is bounded above by $C \cdot \varepsilon^{-\dim \Omega/2} - 1$ for all $0 < \varepsilon \leqslant \varepsilon_0$.

The proof of the lemma now follows easily. Let $d_0 = (c^{**} - c_{**})/C$, and assume that every two consecutive eigenvalues of A_ε in $[\lambda_{**}(\varepsilon), \lambda^{**}(\varepsilon)]$ are strictly less than distance $d_0 \cdot \varepsilon^{\dim \Omega/2}$ apart. This implies $\lambda^{**}(\varepsilon) - \lambda_{**}(\varepsilon) < d_0 \cdot \varepsilon^{\dim \Omega/2} \cdot C \cdot \varepsilon^{-\dim \Omega/2}$, which contradicts the fact that $\lambda^{**}(\varepsilon) - \lambda_{**}(\varepsilon) \geqslant c^{**} - c_{**} = d_0 \cdot C$. □

The above lemma is used to define the decomposition of the phase space $\mathbb{X}$, which is necessary for establishing assumption (H2). Choose constants

$$\underline{c}^{--} < \bar{c}^{--} \ll 0 \ll \underline{c}^{-} < \bar{c}^{-} < \underline{c}^{+} < \bar{c}^{+} < \lambda^{+}_{\max}, \tag{25}$$

such that the differences $\bar{c}^{--} - \underline{c}^{--}$, $\bar{c}^{-} - \underline{c}^{-}$, and $\bar{c}^{+} - \underline{c}^{+}$ are small. The following result is an easy consequence of Lemma 5.1—and leads directly to defining the decomposition of the phase space $\mathbb{X}$.

Corollary 5.2 (Decomposition of the Spectrum). *Suppose that both Assumption* 2.1 *and* 2.3 *are satisfied. Then there exist intervals*

$$J^{--}_{\varepsilon} = [a^{--}_{\varepsilon}, b^{--}_{\varepsilon}] \subset [\underline{c}^{--}, \bar{c}^{--}],$$

$$J^{-}_{\varepsilon} = [a^{-}_{\varepsilon}, b^{-}_{\varepsilon}] \subset [\underline{c}^{-}, \bar{c}^{-}],$$

$$J^{+}_{\varepsilon} = [a^{+}_{\varepsilon}, b^{+}_{\varepsilon}] \subset [\underline{c}^{+}, \bar{c}^{+}],$$

and an ε-independent constant $d>0$ such that for sufficiently small $\varepsilon>0$ the following holds. The intervals J^{--}_{ε}, J^{-}_{ε}, and J^{+}_{ε} are contained in the resolvent set of A_{ε}. Each of them has length at least $d \cdot \varepsilon^{\dim \Omega/2}$, and each component of the complement of their union contains at least one eigenvalue of A_{ε}. Finally, the largest eigenvalue of A_{ε} less than a^{--}_{ε} is at least distance $d \cdot \varepsilon^{\dim \Omega/2}$ from a^{--}_{ε}.

Proof. First of all, apply Lemma 5.1 three times with the interval $[c_*, c^*]$ being chosen as $[\underline{c}^{--}, \bar{c}^{--}]$, $[\underline{c}^{-}, \bar{c}^{-}]$, or $[\underline{c}^{+}, \bar{c}^{+}]$, respectively. This furnishes corresponding ε-dependent intervals $[\hat{a}^{--}_{\varepsilon}, b^{--}_{\varepsilon}]$, $[a^{-}_{\varepsilon}, b^{-}_{\varepsilon}]$, and $[a^{+}_{\varepsilon}, b^{+}_{\varepsilon}]$, as well as a (smallest) common constant d_0. If we now define $a^{--}_{\varepsilon} = (\hat{a}^{--}_{\varepsilon} + b^{--}_{\varepsilon})/2$ and let $d = d_0/2$, then the proof of the corollary is complete. □

Definition 5.3 (Decomposition of the Phase Space). In the situation of Corollary 5.2, define the intervals $I^{--}_{\varepsilon} = (-\infty, a^{--}_{\varepsilon})$, $I^{-}_{\varepsilon} = (b^{--}_{\varepsilon}, a^{-}_{\varepsilon})$, $I^{+}_{\varepsilon} = (b^{-}_{\varepsilon}, a^{+}_{\varepsilon})$, and $I^{++}_{\varepsilon} = (b^{+}_{\varepsilon}, \lambda^{+}_{\max}]$, and let $\mathbb{X}^{--}_{\varepsilon}$, $\mathbb{X}^{-}_{\varepsilon}$, $\mathbb{X}^{+}_{\varepsilon}$, and $\mathbb{X}^{++}_{\varepsilon}$ denote the span of all eigenfunctions of the operator A_{ε} corresponding to eigenvalues in the intervals I^{--}_{ε}, I^{-}_{ε}, I^{+}_{ε}, and I^{++}_{ε}, respectively.

According to Proposition 3.7, the operator $A_{\varepsilon} : \mathbb{X} \to \mathbb{X}$ generates an analytic semigroup $S_{\varepsilon}(t)$, $t \geqslant 0$, on $\mathbb{X}$. The next proposition describes the asymptotic behavior of $S_{\varepsilon}(t)$ on the subspaces defined above.

Proposition 5.4 (Dichotomy Estimates). *Assume that Assumptions* 2.1 *and* 2.3 *are satisfied, and let $A_{\varepsilon} : \mathbb{X} \to \mathbb{X}$ be as in Definition* 3.6. *Let $S_{\varepsilon}(t)$, $t \geqslant 0$, denote the analytic*

semigroup on $\mathbb{X}$ generated by A_ε, and let $\mathbb{X}^\alpha = \mathbb{H}^{2\alpha}_*(\Omega)$ be the fractional power space of Proposition 4.3, *equipped with the norm $||\cdot||_*$.*

Then, using the notation and definitions in Corollary 5.2 *and Definition* 5.3, *for all sufficiently small $\varepsilon > 0$ the following hold*:

(a) *The spaces $\mathbb{X}^-_\varepsilon$, $\mathbb{X}^+_\varepsilon$, and $\mathbb{X}^{++}_\varepsilon$ are finite-dimensional subspaces of $\mathbb{X}^\alpha$ with dimensions proportional to $\varepsilon^{-\dim\Omega/2}$. Furthermore, all of the spaces introduced in Definition* 5.3 *are invariant under $S_\varepsilon(t)$, and we denote the restrictions of the semigroup $S_\varepsilon(t)$ to these spaces by the appropriate superscripts.*

(b) *The following estimates are satisfied for arbitrary $U^{++} \in \mathbb{X}^{++}_\varepsilon$, $U^+ \in \mathbb{X}^+_\varepsilon$, $U^- \in \mathbb{X}^-_\varepsilon$, and $U^{--}_* \in \mathbb{X}^{--}_\varepsilon \cap \mathbb{X}^\alpha$:*

$$||S^{++}_\varepsilon(t)U^{++}||_* \leqslant e^{b^+_\varepsilon t}\cdot||U^{++}||_* \quad \text{for } t\leqslant 0,$$

$$||S^{+}_\varepsilon(t)U^{+}||_* \leqslant e^{a^+_\varepsilon t}\cdot||U^{+}||_* \quad \text{for } t\geqslant 0,$$

$$||S^{+}_\varepsilon(t)U^{+}||_* \leqslant e^{b^-_\varepsilon t}\cdot||U^{+}||_* \quad \text{for } t\leqslant 0,$$

$$||S^{-}_\varepsilon(t)U^{-}||_* \leqslant e^{a^-_\varepsilon t}\cdot||U^{-}||_* \quad \text{for } t\geqslant 0,$$

$$||S^{-}_\varepsilon(t)U^{-}||_* \leqslant e^{b^{--}_\varepsilon t}\cdot||U^{-}||_* \quad \text{for } t\leqslant 0,$$

$$||S^{--}_\varepsilon(t)U^{--}_*||_* \leqslant e^{a^{--}_\varepsilon t}\cdot||U^{--}_*||_* \quad \text{for } t\geqslant 0.$$

There exists a constant $M^{--}_\varepsilon > 0$ such that for $U^{--} \in \mathbb{X}^{--}_\varepsilon$,

$$||S^{--}_\varepsilon(t)U^{--}||_* \leqslant M^{--}_\varepsilon \cdot t^{-\alpha}\cdot e^{a^{--}_\varepsilon t}\cdot||U^{--}||_{\mathbb{L}^2(\Omega)} \quad \text{for } t>0. \tag{26}$$

Moreover, for some ε-independent constant $C>0$ we have

$$M^{--}_\varepsilon \leqslant C\cdot\varepsilon^{-\alpha\cdot(2+\dim\Omega)/2} \quad \text{as } \varepsilon\to 0.$$

Notice that due to the finite dimensions of $\mathbb{X}^-_\varepsilon$, $\mathbb{X}^+_\varepsilon$, and $\mathbb{X}^{++}_\varepsilon$ the linear semigroups $S^-_\varepsilon(t)$, $S^+_\varepsilon(t)$, and $S^{++}_\varepsilon(t)$ can be extended to groups.

(c) *There exists a constant $M_{\alpha,\varepsilon} \geqslant 1$ which is proportional to $\varepsilon^{-\alpha}$ as $\varepsilon\to 0$, as well as an ε-independent constant $C>0$ such that for all $U \in \mathbb{X}^-_\varepsilon \oplus \mathbb{X}^+_\varepsilon \oplus \mathbb{X}^{++}_\varepsilon$ we have*

$$C\cdot||U||_{\mathbb{L}^2(\Omega)} \leqslant ||U||_* \leqslant M_{\alpha,\varepsilon}\cdot||U||_{\mathbb{L}^2(\Omega)}.$$

Proof. The statements in (a) follow easily from (12), Lemma 3.4, Corollary 5.2, and Definition 5.3. As for the proof of part (b), consider an arbitrary $U \in \mathbb{X}$ in form (17).

Then the image of U under the semigroup $S_\varepsilon(t)$ is given by

$$S_\varepsilon(t)U = \sum_{k=1}^{\infty} (e^{\lambda^+_{k,\varepsilon}\cdot t}\cdot \alpha_k^+ \cdot V^+_{k,\varepsilon} + e^{\lambda^-_{k,\varepsilon}\cdot t}\cdot \alpha_k^- \cdot V^-_{k,\varepsilon})\cdot \phi_k \quad \text{for all } t>0.$$

In combination with the definition of $||\cdot||_*$ in (18) and Definition 5.3 this immediately furnishes the first six estimates.

In order to prove (26), let $U\in \mathbb{X}^{--}_\varepsilon$ be arbitrary. Again we assume that the function U is given as the Fourier series (17). For $\eta>0$ one can verify the inequality $e^{-\eta t}\leqslant \eta^{-2\alpha}\cdot(2\alpha)^{2\alpha}/(e^{2\alpha}\cdot t^{2\alpha})$ for all $t\geqslant 0$. Notice also that whenever $\alpha_k^\pm\neq 0$, we must have $a^{--}_\varepsilon - \lambda^\pm_{k,\varepsilon}>0$. Thus, applying the inequality with $\eta = 2(a^{--}_\varepsilon - \lambda^\pm_{k,\varepsilon})$, we get that for all $t>0$

$$\begin{aligned} ||S^{--}_\varepsilon(t)U||^2_* &= \sum_{k=1}^{\infty}(1+\kappa_k)^{2\alpha}\cdot(e^{2\lambda^+_{k,\varepsilon}\cdot t}\cdot(\alpha_k^+)^2 + e^{2\lambda^-_{k,\varepsilon}\cdot t}\cdot(\alpha_k^-)^2) \\ &\leqslant \sum_{k=1}^{\infty}\left(\frac{(1+\kappa_k)^{2\alpha}\cdot(\alpha_k^+)^2}{(a^{--}_\varepsilon-\lambda^+_{k,\varepsilon})^{2\alpha}} + \frac{(1+\kappa_k)^{2\alpha}\cdot(\alpha_k^-)^2}{(a^{--}_\varepsilon-\lambda^-_{k,\varepsilon})^{2\alpha}}\right) \\ &\quad \cdot \frac{\alpha^{2\alpha}}{e^{2\alpha}\cdot t^{2\alpha}}\cdot e^{2a^{--}_\varepsilon\cdot t}. \end{aligned}$$

Assume for the moment that there exists an ε-independent constant $C>0$ such that

$$\frac{1+\kappa_k}{a^{--}_\varepsilon - \lambda^\pm_{k,\varepsilon}} \leqslant C\cdot\varepsilon^{-(2+\dim\Omega)/2} \quad \text{whenever } \lambda^\pm_{k,\varepsilon} < a^{--}_\varepsilon. \tag{27}$$

Then we can continue the previous estimate and obtain for all $t>0$ the estimate

$$\begin{aligned} ||S^{--}_\varepsilon(t)U||^2_* &\leqslant \left(\frac{C\cdot\alpha}{e}\right)^{2\alpha}\cdot\varepsilon^{-\alpha\cdot(2+\dim\Omega)}\cdot t^{-2\alpha}\cdot e^{2a^{--}_\varepsilon\cdot t}\cdot\sum_{k=1}^{\infty}((\alpha_k^+)^2+(\alpha_k^-)^2) \\ &= \left(\frac{C\cdot\alpha}{e}\right)^{2\alpha}\cdot\varepsilon^{-\alpha\cdot(2+\dim\Omega)}\cdot t^{-2\alpha}\cdot e^{2a^{--}_\varepsilon\cdot t}\cdot||T_\varepsilon U||^2_{\mathbb{L}^2(\Omega)}, \end{aligned}$$

where $T_\varepsilon:\mathbb{X}\to\mathbb{X}$ was defined in (19). Together with (20) this furnishes the seventh estimate of (b) with $M^{--}_\varepsilon = c_2(C\alpha/e)^\alpha\cdot\varepsilon^{-\alpha\cdot(2+\dim\Omega)/2}$.

In order to complete the proof of (b) we still have to verify (27). Let s_r be the larger zero of the function $\lambda^+(\cdot)$ as in Lemma 3.4 and Fig. 3. Let $s_\#>s_r$ be the unique number such that $\lambda^+(s_\#) = \underline{c}^{--}-1$. Then according to the asymptotic behavior of $\lambda^\pm(\cdot)$ derived in Lemma 3.4 there exists an ε-independent constant C such that

$$0<\frac{1+s}{\underline{c}^{--}-\lambda^\pm(s)}\leqslant C \quad \text{for all } s\geqslant s_\#.$$

Fix $0<\varepsilon\leqslant 1$. For $\varepsilon\kappa_k \geqslant s_\#$ (which furnishes $\lambda^{\pm}_{k,\varepsilon} < a^{--}_\varepsilon$) we obtain the estimate:

$$\frac{1+\kappa_k}{a^{--}_\varepsilon - \lambda^{\pm}_{k,\varepsilon}} \leqslant \frac{1+\varepsilon\kappa_k}{\underline{c}^{--} - \lambda^{\pm}(\varepsilon\kappa_k)} \cdot \frac{1+\kappa_k}{1+\varepsilon\kappa_k} \leqslant C \cdot \frac{1}{\varepsilon} \leqslant C \cdot \varepsilon^{-(2+\dim\Omega)/2}.$$

Now assume $\varepsilon\kappa_k < s_\#$ and that $\lambda^{\pm}_{k,\varepsilon} < a^{--}_\varepsilon$. According to Corollary 5.2 we then have $\lambda^{\pm}_{k,\varepsilon} \leqslant a^{--}_\varepsilon - d\cdot\varepsilon^{\dim\Omega/2}$. This furnishes

$$\frac{1+\kappa_k}{a^{--}_\varepsilon - \lambda^{\pm}_{k,\varepsilon}} = \frac{1+\varepsilon\kappa_k}{a^{--}_\varepsilon - \lambda^{\pm}_{k,\varepsilon}} \cdot \frac{1+\kappa_k}{1+\varepsilon\kappa_k} \leqslant \frac{1+s_\#}{d\cdot\varepsilon^{\dim\Omega/2}} \cdot \frac{1}{\varepsilon},$$

which completes the verification of (27).

As for part (c), let $U \in \mathbb{X}^-_\varepsilon \oplus \mathbb{X}^+_\varepsilon \oplus \mathbb{X}^{++}_\varepsilon$ be arbitrary. If we assume that U is given as in (17), then (18) and (20) imply

$$c_1^2 \cdot ||U||^2_{\mathbb{L}^2(\Omega)} \leqslant ||T_\varepsilon U||^2_{\mathbb{L}^2(\Omega)} = \sum_{k=1}^{\infty} ((\alpha^+_k)^2 + (\alpha^-_k)^2) \leqslant ||U||^2_*.$$

Furthermore, if we choose $s_\#$ as above, then $\varepsilon\kappa_k \geqslant s_\#$ implies $\alpha^{\pm}_k = 0$ in the series representation (17) of $U \in \mathbb{X}^-_\varepsilon \oplus \mathbb{X}^+_\varepsilon \oplus \mathbb{X}^{++}_\varepsilon$. Thus, (18) and (20) imply

$$||U||^2_* \leqslant \left(1+\frac{s_\#}{\varepsilon}\right)^{2\alpha} \cdot ||T_\varepsilon U||^2_{\mathbb{L}^2(\Omega)} \leqslant c_2^2 \cdot \left(1+\frac{s_\#}{\varepsilon}\right)^{2\alpha} \cdot ||U||^2_{\mathbb{L}^2(\Omega)},$$

and (c) follows with $M_{\alpha,\varepsilon} = c_2 \cdot (s_\# + \varepsilon)^\alpha \cdot \varepsilon^{-\alpha}$. □

5.2. Estimates for the nonlinearity

In this section we verify hypothesis (H3) which was stated in the beginning of Section 5. The following lemma shows that the nonlinearity F in (23) is a continuously differentiable function from $\mathbb{X}^\alpha$ to $\mathbb{X}$, provided α is sufficiently large.

Lemma 5.5 (Properties of F). *Suppose that Assumptions* 2.1–2.3 *are satisfied, and let h be defined as in* (24). *Furthermore, for arbitrary $U = (u,v) \in \mathbb{X}^\alpha$ let $F(U) = h(u,v)$. Then for every α satisfying* $\dim\Omega/4 < \alpha < 1$ *this defines a nonlinear mapping $F : \mathbb{X}^\alpha \to \mathbb{X}$ which is continuously Fréchet differentiable. Furthermore, there exist positive constants C and R_0 such that for any $0<R\leqslant R_0$ the following holds. For arbitrary $U, V \in \mathbb{X}^\alpha$ with*

$$||U - (\bar{u}_0, \bar{v}_0)||_* \leqslant R \quad \text{and} \quad ||V - (\bar{u}_0, \bar{v}_0)||_* \leqslant R,$$

we have

$$||F(U) - F(V)||_{\mathbb{X}} \leqslant C \cdot R^\sigma \cdot ||U - V||_*.$$

Proof. Due to $\dim\Omega/4<\alpha<1$ we can apply Lemma 4.4, which shows that every $U\in\mathbb{X}^\alpha$ is almost everywhere equal to a continuous function on $\bar{\Omega}$. Thus, the function $F(U)$ can be defined pointwise, is bounded on Ω, and therefore an element of the space $\mathbb{X}=\mathbb{L}^2(\Omega)$. Standard results on Nemitskii operators then imply that the nonlinear operator $F:\mathbb{X}^\alpha\to\mathbb{X}$ is continuously differentiable with Fréchet derivative

$$DF(U)[H]=Dh(u,v)H \quad \text{for all } U=(u,v)\in\mathbb{X}^\alpha \text{ and } H\in\mathbb{X}^\alpha.$$

(See for example [8, Section 4.3.1] for similar considerations in a one-dimensional setting. Using Hale's approach, one can easily compute the Gateaux derivative of F. Together with Zeidler [28, Proposition 4.8(c)] this establishes the Frechet differentiability, once the continuity of the Gateaux derivative has been verified.)

Because of Assumption 2.2 there exists a neighborhood of $(\bar{u}_0,\bar{v}_0)$ in $\mathbb{R}^2$ such that for all $(u,v)\in\mathbb{R}^2$ in this neighborhood we have $||Dh(u,v)||\leqslant C\cdot||(u,v)-(\bar{u}_0,\bar{v}_0)||^\sigma$. Thus, there exist positive constants R_0 and C such that for all $W\in\mathbb{X}^\alpha$ we have

$$||DF(W)||_{\mathbb{X}^\alpha,\mathbb{X}}\leqslant C\cdot||W-(\bar{u}_0,\bar{v}_0)||_*^\sigma \quad \text{for } ||W-(\bar{u}_0,\bar{v}_0)||_*\leqslant R_0,$$

again due to Lemma 4.4. Together with the mean value theorem for Fréchet differentiable functions (see [28]) this completes the proof of the lemma. □

5.3. *The main result on the early stages*

Finally, we are in a position to describe the early pattern formation stages in the nonlinear parabolic system (4). As mentioned before, we will show that the behavior of most solutions of (4) originating near the homogeneous equilibrium $(\bar{u}_0,\bar{v}_0)$ is determined by a dominating subspace $\mathscr{Y}_\varepsilon^+$, and that functions in this subspace exhibit the complicated geometries described in the introduction. The following definition introduces the dominating subspace in more detail.

Definition 5.6 (Dominating Subspace $\mathscr{Y}_\varepsilon^+$). Assume the situation of Definition 5.3. Then we define the dominating subspace $\mathscr{Y}_\varepsilon^+$ by

$$\mathscr{Y}_\varepsilon^+=\mathbb{X}_\varepsilon^+\oplus\mathbb{X}_\varepsilon^{++}.$$

Due to Proposition 5.4 its dimension is proportional to $\varepsilon^{-\dim\Omega/2}$ as $\varepsilon\to0$.

The geometry of functions in the dominating subspace $\mathscr{Y}_\varepsilon^+$ has been described by Maier-Paape and Wanner [14]. In Section 4 of their paper it is shown that these functions exhibit a common wavelength which is of the order of $\varepsilon^{1/2}$. (Notice that in the application to the Cahn–Hilliard equation our parameter ε is replaced by ε^2. This is the only necessary change for applying the results of [14] to our situation.) More precisely, let $\phi\in\mathscr{Y}_\varepsilon^+$ be arbitrary. Then the level set $\{x\in\Omega:\phi(x)=0\}$ divides Ω into a set where ϕ is positive, and a set where ϕ is negative. These two sets are called

nodal domains. Furthermore, let $x_0 \in \Omega$ denote a "typical" point (see [14] for a precise definition of this notion), and let $G \subset \Omega$ denote the nodal domain of ϕ containing x_0. Then for any ball contained in G with radius r and center x_0 we have $r \leqslant C \cdot \varepsilon^{1/2}$.

We are finally in a position to formally restate and prove Theorem 2.5. This is analogous to Fig. 4 and [15, Section 3.4].

Theorem 5.7. *Consider the reaction–diffusion system* (4) *and assume that Assumptions* 2.1–2.3 *are satisfied. Furthermore, assume that* $\dim \Omega/4 < \alpha < 1$, *let* $\mathbb{X}^\alpha$ *be as in Proposition* 4.3, *and choose and fix* $0 < p \ll 1$ *and* $0 < d_0 \ll 1$. *Then there exists a positive constant* ε_0, *so that for all* $0 < \varepsilon \leqslant \varepsilon_0$ *there exist* $0 < r_\varepsilon < R_\varepsilon$ *such that with* $\rho_\varepsilon = d_0 \cdot R_\varepsilon$ *the following is true*:

(a) *The constants* r_ε *and* R_ε *are proportional to* $\varepsilon^{(2\alpha+\dim\Omega)/(2\sigma)}$ *as* $\varepsilon \to 0$.

(b) *The ball* $\mathbb{B}_{R_\varepsilon}(\bar{u}_0, \bar{v}_0) \subset \mathbb{X}^\alpha$ *contains a finite-dimensional inertial manifold* $\mathcal{N}_\varepsilon$ *passing through* $(\bar{u}_0, \bar{v}_0)$ *which exponentially attracts all solutions of* (4) *originating near the homogeneous equilibrium. Furthermore, the manifold* $\mathcal{N}_\varepsilon$ *is of class* C^1; *it is tangent to* $\mathbb{X}_\varepsilon^- \oplus \mathbb{X}_\varepsilon^+ \oplus \mathbb{X}_\varepsilon^{++}$ at $(\bar{u}_0, \bar{v}_0)$, *and it carries a natural Lebesgue measure induced by this tangent space.*

(c) *Let* $\mathbb{M}_{r_\varepsilon}$ *denote the set of all initial conditions in* $\mathcal{N}_\varepsilon \cap \mathbb{B}_{r_\varepsilon}(\bar{u}_0, \bar{v}_0)$ *whose corresponding solution of* (4) *either remains in the larger ball* $\mathbb{B}_{R_\varepsilon}(\bar{u}_0, \bar{v}_0)$ *for all positive time, or has distance greater than* ρ_ε *from* $(\bar{u}_0, \bar{v}_0) + \mathcal{Y}_\varepsilon^+$ *upon exiting* $\mathbb{B}_{R_\varepsilon}(\bar{u}_0, \bar{v}_0)$, *i.e., the set* $\mathbb{M}_{r_\varepsilon}$ *corresponds to all solutions originating close to the homogeneous equilibrium which are not dominated by* $\mathcal{Y}_\varepsilon^+$. *Then*

$$\frac{vol(\mathbb{M}_{r_\varepsilon})}{vol(\mathbb{B}_{r_\varepsilon}(\bar{u}_0, \bar{v}_0) \cap \mathcal{N}_\varepsilon)} \leqslant p,$$

where vol denotes the canonical Lebesgue measure on $\mathcal{N}_\varepsilon$.

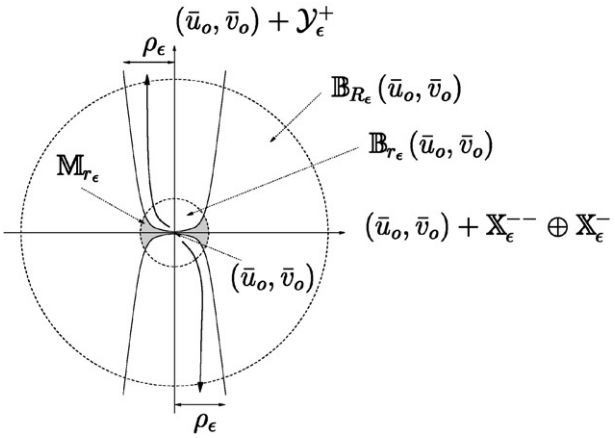

Fig. 4. The early stages of the pattern formation process.

Proof. In order to apply the abstract theory of Section 2 in [15], one has to verify hypotheses (H1)–(H3) mentioned in the beginning of Section 5. They have been verified in Sections 5.1 and 5.2. The size of the radii r_ε and R_ε can be determined as in [15, Remark 3.1].

As it stands, the theory of Section 2 in [15] cannot be applied directly. This is due to the fact that our operator A_ε is not self-adjoint, and therefore the spaces introduced in Definition 5.3 are not pairwise orthogonal. However, an inspection of the proofs in [15] shows that this orthogonality is not necessary. It suffices to assume that the angle between any two of the subspaces in Definition 5.3 is bounded away from zero for all $0<\varepsilon\leqslant\varepsilon_0$, since this implies the boundedness of the corresponding projection operators. This lower bound on the angle is obtained using Lemma 3.5. □

6. Almost linear behavior

As mentioned in Section 1, the results of the last section only describe the first stage of the pattern formation process. At the end of this stage, most solutions of the reaction–diffusion system (4) originating near the homogeneous equilibrium $(\bar u_0,\bar v_0)$ leave the ball $\mathbb{B}_{R_\varepsilon}(\bar u_0,\bar v_0)$ close to the dominating subspace $\mathscr{Y}_\varepsilon^+$. We will show in this section that in the subsequent pattern formation stage, they will closely follow the corresponding solutions of the linearized equation, even though they are not near equilibrium. This is because at the end of the first stage, most solutions enter a region of phase space in which the effect of the nonlinearity is small.

Specifically, we apply Lemma 2.8 and Theorem 2.10 from Sander and Wanner [24] to bound the *relative distance* $||U-(\bar u_0,\bar v_0)-U_{\mathrm{lin}}||_*/||U_{\mathrm{lin}}||_*$ between the solutions U and U_{lin} to the nonlinear and linearized equations with initial conditions U_0 near $(\bar u_0,\bar v_0)$ and $U_0-(\bar u_0,\bar v_0)$ near $(0,0)$, respectively.

Lemma 2.8 in [24] shows that for arbitrary small constants δ_0 and q, a solution to the nonlinear equation (23) starting in a cone of angle δ_0 stays in a slightly larger cone of angle δ (depending on δ_0 and q) as long as the relative distance is less than q.

Theorem 2.10 in [24] is as follows: Fix a relative distance ζ. Let $T_{\max}$ be the maximal time such that the linear and nonlinear solutions exist, that the operator generating the semigroup satisfies certain estimates, and that there is a small Lipschitz bound on the nonlinearity. Let R_0 be a sufficiently small constant depending on ζ, and let $R_1>R_0$ be a constant depending on R_0 and ζ. Finally, let the initial condition U_0 have $*$-norm between R_0 and R_1. Then until the solution reaches $*$-norm R_1 (or reaches time $T_{\max}$), the relative distance is less than ζ.

In Section 6.1, we set up conditions such that A_ε satisfies the appropriate semigroup estimates Theorem 2.10 in [24]. In Section 6.2, we define a cone around the dominating subspace and show that in this cone the nonlinearity satisfies the Lipschitz estimates of Theorem 2.10 in [24]. In Section 6.3, we state and prove our main result. The result gives the specific ε-dependence of R_0, R_1, and ζ. Further, we use Lemma 2.8 in [24] to show that radius R_1 is achieved before time $T_{\max}$ is reached.

6.1. Linear estimates

In this section, we derive estimates on the growth of the linear semigroup analogous to the results obtained in Proposition 5.4.

Proposition 6.1 (Linear Growth). *Assume that Assumptions* 2.1 *and* 2.3 *are satisfied, and let* $A_\varepsilon : \mathbb{X} \to \mathbb{X}$ *be as in Definition* 3.6. *Let* $S_\varepsilon(t)$, $t \geqslant 0$, *denote the analytic semigroup on* $\mathbb{X}$ *generated by* A_ε, *and let* $\mathbb{X}^\alpha = \mathbb{H}^{2\alpha}_*(\Omega)$ *be the fractional power space of Proposition* 4.3, *equipped with the norm* $||\cdot||_*$. *Furthermore, let* $\beta > \lambda^+_{\max}$ *be an* ε*-independent constant, with* $\lambda^+_{\max}$ *as in Lemma* 3.4. *Then there exists an* ε*-independent constant* $C > 0$ *such that for all* $0 < \varepsilon \leqslant 1$ *we have*

$$||S_\varepsilon(t)U||_* \leqslant C \cdot \varepsilon^{-\alpha} \cdot t^{-\alpha} \cdot e^{\beta \cdot t} \cdot ||U||_{\mathbb{L}^2(\Omega)} \quad \text{for } t > 0, \quad U \in \mathbb{X},$$

$$||S_\varepsilon(t)U||_* \leqslant e^{\lambda^+_{\max} \cdot t} \cdot ||U||_* \quad \text{for } t \geqslant 0, \quad U \in \mathbb{X}^\alpha.$$

Proof. The proof of the above estimates follows easily using an argument similar to the proof of the last inequality in Proposition 5.4(b). In this case, replace estimate (27) by

$$\frac{1 + \kappa_k}{\beta - \lambda^\pm_{k,\varepsilon}} = \frac{1 + \varepsilon\kappa_k}{\beta - \lambda^\pm(\varepsilon\kappa_k)} \cdot \frac{\varepsilon + \varepsilon\kappa_k}{1 + \varepsilon\kappa_k} \cdot \varepsilon^{-1} \leqslant C \cdot \varepsilon^{-1}.$$

The proof of the proposition follows. □

6.2. Estimates for the nonlinearity

The main ingredient for describing the second stage of the pattern formation process is the fact that the influence of the nonlinearity remains extremely small in certain regions of phase space. In other words, the bound obtained in Lemma 5.5 can be improved considerably if we are close to the dominating subspace $\mathscr{Y}^+_\varepsilon$. In order to describe this region in more detail, we use Definitions 5.3 and 5.6 once again and define

$$\mathscr{Y}^+_\varepsilon = \mathbb{X}^+_\varepsilon \oplus \mathbb{X}^{++}_\varepsilon \subset \mathbb{X}^\alpha, \quad \mathscr{Y}^-_\varepsilon = (\mathbb{X}^{--}_\varepsilon \cap \mathbb{X}^\alpha) \oplus \mathbb{X}^-_\varepsilon \subset \mathbb{X}^\alpha. \tag{28}$$

With respect to this decomposition we consider cones $(\bar{u}_0, \bar{v}_0) + \mathscr{K}_\delta \subset \mathbb{X}^\alpha$, where

$$\mathscr{K}_\delta = \{U \in \mathbb{X}^\alpha : ||U_-||_* \leqslant \delta \cdot ||U_+||_*, U = U_+ + U_- \in \mathscr{Y}^+_\varepsilon \oplus \mathscr{Y}^-_\varepsilon\} \tag{29}$$

for $\delta > 0$. The following proposition shows that on these translated cones the nonlinearity remains small, even relatively far from the homogeneous equilibrium.

Proposition 6.2 (Smallness of F). *Suppose that Assumptions* 2.1–2.3 *are satisfied, and let h be defined as in* (24). *For $U=(u,v)\in\mathbb{X}^\alpha$ define $F(U)=h(u,v)$. Finally, let* $\dim\Omega/4<\alpha<1$ *and $\delta_0>0$ be arbitrary, and set*

$$\delta_\varepsilon=\delta_0\cdot\varepsilon^{\alpha-\dim\Omega/4}. \tag{30}$$

Then there exist ε-independent constants $M_1, M_2>0$ such that for every $0<\varepsilon\leqslant 1$ and every $U\in\mathscr{K}_{\delta_\varepsilon}$ with

$$||U||_*\leqslant M_1\cdot\varepsilon^{-\alpha+\dim\Omega/4} \tag{31}$$

we have

$$||F((\bar u_0,\bar v_0)+U)||_{\mathbb{L}^2(\Omega)}\leqslant M_2\cdot\varepsilon^{(\alpha-\dim\Omega/4)\cdot(\sigma+1)}\cdot||U||_*^{\sigma+1}. \tag{32}$$

The constants M_1 and M_2 depend only on h, δ_0, and Ω.

Proof. According to Assumption 2.2 and the definition of $h:\mathbb{R}^2\to\mathbb{R}^2$ in (24) there exist constants $\tilde M_1>0$ and $\tilde M_2>0$ such that for $(u,v)\in\mathbb{R}^2$ we have

$$||h((\bar u_0,\bar v_0)+(u,v))||_{\mathbb{R}^2}\leqslant\tilde M_1\cdot||(u,v)||_{\mathbb{R}^2}^{\sigma+1}\quad\text{for all } ||(u,v)||_{\mathbb{R}^2}\leqslant\tilde M_2. \tag{33}$$

Furthermore, we verify below that for δ_ε as in (30) there exists a constant $\tilde M_3>0$ such that for all $0<\varepsilon\leqslant 1$ and arbitrary $U\in\mathscr{K}_{\delta_\varepsilon}$ we have

$$||U||_{C(\bar\Omega)}\leqslant\tilde M_3\cdot\varepsilon^{\alpha-\dim\Omega/4}\cdot||U||_*. \tag{34}$$

Now let $0<\varepsilon\leqslant 1$ be arbitrary, define $M_1=\tilde M_2/\tilde M_3$, and let $U\in\mathscr{K}_{\delta_\varepsilon}$ be such that (31) is satisfied. Then (34) implies $||U||_{C(\bar\Omega)}\leqslant\tilde M_2$, and (33) furnishes

$$\begin{aligned}||h((\bar u_0,\bar v_0)+U)||_{C(\bar\Omega)}&\leqslant\tilde M_1\cdot||U||_{C(\bar\Omega)}^{\sigma+1}\\&\leqslant\tilde M_1\tilde M_3^{\sigma+1}\cdot\varepsilon^{(\alpha-\dim\Omega/4)\cdot(\sigma+1)}\cdot||U||_*^{\sigma+1}.\end{aligned}$$

This immediately implies the desired estimate (32).

In order to complete the proof of the proposition we only have to verify (34). Let $U\in\mathscr{K}_{\delta_\varepsilon}$ be arbitrary, and let $U=U_+ + U_-\in\mathscr{Y}_\varepsilon^+\oplus\mathscr{Y}_\varepsilon^-$. Then U_+ is given as a finite sum of the form

$$U_+=\sum_{k=k_1}^{k_2}\alpha_k^+\cdot V_{k,\varepsilon}^+\cdot\phi_k,$$

where $V_{k,\varepsilon}^+\in\mathbb{R}^2$ and $\phi_k\in L^2(\Omega)$ with $||\phi_k||_{L^2(\Omega)}=1$ as in (17). Using Assumption 3.2, the maximum norms of the $L^2(\Omega)$-normalized eigenfunctions of the Laplacian are uniformly bounded, and followed by Hölder's inequality for sequences, this

immediately furnishes

$$||U_+||_{C(\bar\Omega)} \leqslant C \cdot \underbrace{\left(\sum_{k=k_1}^{k_2} (1+\kappa_k)^{2\alpha} \cdot (\alpha_k^+)^2 \right)^{1/2}}_{=||U_+||_*} \cdot \left(\sum_{k=k_1}^{k_2} \frac{1}{(1+\kappa_k)^{2\alpha}} \right)^{1/2}. \tag{35}$$

Lemma 3.4 and Proposition 3.7, in combination with Definition 5.6, show that κ_{k_1} is bounded below by s_ℓ/ε as $\varepsilon \to 0$. Together with the asymptotic growth of the κ_k given in (12) we therefore obtain the following estimates. Note that in these estimates and throughout this proof C indicates a constant, but not all C's represent the same constant.

$$\begin{aligned} \sum_{k=k_1}^{k_2} \frac{1}{(1+\kappa_k)^{2\alpha}} &\leqslant C \cdot \sum_{k=k_1}^{k_2} k^{-4\alpha/\dim\Omega} \leqslant C \cdot \int_{k_1-1}^{k_2} \tau^{-4\alpha/\dim\Omega}\, d\tau \\ &\leqslant C \cdot (k_1-1)^{1-4\alpha/\dim\Omega} \leqslant C \cdot \varepsilon^{(4\alpha-\dim\Omega)/2}, \end{aligned}$$

and in view of (35)

$$||U_+||_{C(\bar\Omega)} \leqslant C \cdot \varepsilon^{\alpha-\dim\Omega/4} \cdot ||U_+||_*. \tag{36}$$

According to Lemma 4.4 and $U \in \mathscr{K}_{\delta_\varepsilon}$ there exists an ε-independent constant C such that $||U_-||_{C(\bar\Omega)} \leqslant C \cdot ||U_-||_* \leqslant C \cdot \delta_\varepsilon \cdot ||U_+||_*$, and (36) furnishes with (30) the estimate

$$\begin{aligned} ||U||_{C(\bar\Omega)} &\leqslant ||U_+||_{C(\bar\Omega)} + ||U_-||_{C(\bar\Omega)} \leqslant C \cdot \varepsilon^{\alpha-\dim\Omega/4} \cdot ||U_+||_* \\ &\leqslant C \cdot \varepsilon^{\alpha-\dim\Omega/4} \cdot ||U||_*. \end{aligned}$$

For the last estimate we use the fact that the operator norm of the projection onto $\mathscr{Y}_\varepsilon^+$ can be bounded by an ε-independent constant. This completes the proof of the proposition. □

6.3. The main result on the later stages

The result of the previous section shows that the influence of the nonlinearity of (4) remains small up to large distances from the homogeneous equilibrium, provided we stay in a certain cone around the dominating subspace $\mathscr{Y}_\varepsilon^+$. This fact is used in the next result, which describes the second stage of the pattern formation process in (4). The proof of the theorem uses the abstract theory in [24], which was developed there for the case of the Cahn–Hilliard equation.

Theorem 6.3. *Consider the reaction–diffusion system* (4) *and assume that Assumptions* 2.1–2.3 *are satisfied. Assume that* $\dim \Omega/4 < \alpha < 1$, *let* $\mathbb{X}^\alpha$ *be as in Proposition* 4.3, *and choose and fix* $\delta_0 \in (0, \frac{1}{2})$ *and* $0 < \rho \ll 1$. *Then there exist constants* $D > 0$ *and* $0 \ll \underline{c}^- < \bar{c}^- < \lambda^+_{\max}$ *such that for the splitting of* $\mathbb{X}^\alpha$ *defined in* (28) *and Definition* 5.3, *and for all* $\varepsilon \in (0, 1]$ *the following is true. If* $U_0 \in (\bar{u}_0, \bar{v}_0) + \mathcal{K}_{\delta_\varepsilon}$, *with* $\delta_\varepsilon = \delta_0 \cdot \varepsilon^{\alpha - \dim \Omega/4}$, *is any initial condition satisfying*

$$0 < ||U_0 - (\bar{u}_0, \bar{v}_0)||_* \leqslant \min\{1, (D \cdot \varepsilon^{-(\alpha - \dim \Omega/4) + \alpha/\sigma + \rho})^{1/(1-\rho)}\}, \tag{37}$$

and if U *and* U_{lin} *denote the solutions of the nonlinear equation* (4) *and the linearized equation* (11) *originating at* U_0 *and* $U_0 - (\bar{u}_0, \bar{v}_0)$, *respectively, then there exists a time* $T > 0$ *such that the following first occurs*:

$$||U(T) - (\bar{u}_0, \bar{v}_0)||_* = D \cdot \varepsilon^{-(\alpha - \dim \Omega/4) + \alpha/\sigma + \rho} \cdot ||U_0 - (\bar{u}_0, \bar{v}_0)||_*^\rho. \tag{38}$$

For all $t \in [0, T]$ *we have*

$$\frac{||U(t) - (\bar{u}_0, \bar{v}_0) - U_{\text{lin}}(t)||_*}{||U_{\text{lin}}(t)||_*} \leqslant \frac{\delta_0}{2} \cdot \varepsilon^{\alpha - \dim \Omega/4}. \tag{39}$$

Proof. The proof uses the theory developed in [24]. The reader is referred to this reference for more details; see especially the proof of Theorem 3.4 in [24]. Choose a constant $m > 0$ small enough such that

$$\frac{m}{m+\sigma} \leqslant \rho \quad \text{and} \quad \frac{\alpha - \sigma(\alpha - \dim \Omega/4)}{m+\sigma} \leqslant -\left(\alpha - \frac{\dim \Omega}{4}\right) + \frac{\alpha}{\sigma} + \rho. \tag{40}$$

Furthermore, choose the constants $\underline{c}^-$ and $\bar{c}^-$ such that

$$\frac{\lambda^+_{\max}}{m+1} < \underline{c}^- < \bar{c}^- < \lambda^+_{\max}.$$

Using the constants defined in Corollary 5.2, we let $\beta = (\lambda^+_{\max} + (m+1) \cdot \underline{c}^-)/2$ and $\gamma_\varepsilon = (a^-_\varepsilon + b^-_\varepsilon)/2$. Note that $\underline{c}^- < \gamma_\varepsilon < \bar{c}^-$. For the above choice of β, we now denote the factor $C \cdot \varepsilon^{-\alpha}$ in the first estimate of Proposition 6.1 by K_ε. Then there exists an ε-independent constant N_0 such that for all $0 < \varepsilon \leqslant 1$ we have

$$\frac{K_\varepsilon \cdot d(\alpha)}{(\beta - \lambda^+_{\max})^{1-\alpha} \cdot (1-\alpha)} \leqslant N_0 \cdot \varepsilon^{-\alpha},$$

where $d(\alpha)$ is a constant depending only on α, as defined in [24, Lemma 2.4]. Let

$$M_\varepsilon = M_2 \cdot \varepsilon^{(\alpha - \dim \Omega/4) \cdot (\sigma+1)},$$

$$N_\varepsilon = N_0 \cdot \varepsilon^{-\alpha} \cdot M_\varepsilon = N_0 \cdot M_2 \cdot \varepsilon^{\alpha\sigma - (\sigma+1)\dim \Omega/4},$$

with M_2 as in (32), but for the larger cone $\mathscr{K}_{2\delta_\varepsilon}$. Consider the ε-dependent constant

$$R_\varepsilon = \left(\frac{2^{\alpha-1} \cdot ((m+1) \cdot \underline{c}^- - \lambda_{\max}^+)^{1-\alpha}}{K_\varepsilon \cdot M_\varepsilon \cdot \Gamma(1-\alpha)} \right)^{1/\sigma},$$

which is proportional to $\varepsilon^{-(\alpha-\dim \Omega/4)+\dim \Omega/(4\sigma)}$ in the limit $\varepsilon \to 0$. Choose the constant $D>0$ in such a way that the expression $D \cdot \varepsilon^{-(\alpha-\dim \Omega/4)+\alpha/\sigma+\rho}$ is strictly less than R_ε and $(\delta_\varepsilon/(2N_\varepsilon))^{1/(m+\sigma)} \cdot 2^{-m/(m+\sigma)} \cdot (1+\delta_\varepsilon^2)^{-(m+1)/(2m+2\sigma)}$, as well as less than $M_1 \cdot \varepsilon^{-\alpha+\dim \Omega/4}$ in (5) for $\mathscr{K}_{2\delta_\varepsilon}$, for arbitrary $\varepsilon \in (0,1]$. This is possible due to the second estimate in (40). Finally, let $\zeta_\varepsilon = \delta_\varepsilon/2$, and fix an arbitrary initial condition $U_0 \in (\bar{u}_0, \bar{v}_0) + \mathscr{K}_{\delta_\varepsilon}$ satisfying (37).

We now turn to establishing the existence of the time T mentioned in the formulation of the theorem. Since this has to be done in several steps, we outline the remainder of the proof briefly. Our proof is based on Theorem 2.10 in [24]. Since the assumptions of this theorem will not be satisfied for all times $t \geqslant 0$, we first define a (maximal) interval $[0, T_{\max}]$ on which they are met. (The crucial step consists in verifying the validity of (2.12) in [24].) Theorem 2.10 in [24] then furnishes a radius R_1 and an interval $[0, T_1] \subset [0, T_{\max}]$, on which the relative distance between $U - (\bar{u}_0, \bar{v}_0)$ and U_{lin} remains small and the norm of $U - (\bar{u}_0, \bar{v}_0)$ is at most R_1. This radius R_1 is larger than the radius R given by the right-hand side of (38). We therefore denote by $T_0 \leqslant T_1$ the maximal time such that $||U(t) - (\bar{u}_0, \bar{v}_0)||_* < R$ for all $t \in [0, T_0)$. Our theorem is proven as long as $T_0 < T_{\max}$, since this guarantees that the norm of $U - (\bar{u}_0, \bar{v}_0)$ actually reaches R at time T_0. In this case we have $T = T_0$. On the other hand, for $T_0 = T_1 = T_{\max}$ we could not draw this conclusion, and we therefore show separately that this equality is impossible.

The detailed arguments are as follows. Choose $T_{\max} \geqslant 0$ maximal such that for all $t \in [0, T_{\max})$ the solution $U(t)$ is contained in $(\bar{u}_0, \bar{v}_0) + \mathscr{K}_{2\delta_\varepsilon}$ and $||U(t) - (\bar{u}_0, \bar{v}_0)||_* < R_\varepsilon$. Note that by Proposition 6.2 for all $t \in [0, T_{\max})$ we have $||F(U(t))||_{\mathbb{L}^2(\Omega)} \leqslant L_\varepsilon \cdot ||U(t) - (\bar{u}_0, \bar{v}_0)||_*$ with $L_\varepsilon = M_\varepsilon \cdot R_\varepsilon^\sigma$. This, the lower bound on γ_ε, and the definition of $T_{\max}$ imply that estimate (2.12) in [24] holds for all $t \in [0, T_{\max}]$. Notice also that due to (37) we have

$$\begin{aligned} ||U_0 - (\bar{u}_0, \bar{v}_0)||_* &\leqslant D \cdot \varepsilon^{-(\alpha-\dim \Omega/4)+\alpha/\sigma+\rho} \cdot ||U_0 - (\bar{u}_0, \bar{v}_0)||_*^\rho \\ &\leqslant D \cdot \varepsilon^{-(\alpha-\dim \Omega/4)+\alpha/\sigma+\rho} < R_\varepsilon, \end{aligned}$$

which implies $T_{\max} > 0$. Define

$$\begin{aligned} R_1 = ||U_0 - (\bar{u}_0, \bar{v}_0)||_*^{m/(m+\sigma)} \cdot \left(\frac{\delta_\varepsilon}{2N_\varepsilon} \right)^{1/(m+\sigma)} \\ \cdot 2^{-m/(m+\sigma)} \cdot (1+\delta_\varepsilon^2)^{-(m+1)/(2m+2\sigma)}, \end{aligned}$$

and let $T_1 \in [0, T_{\max}]$ be the maximal time such that $||U(t) - (\bar{u}_0, \bar{v}_0)||_* < R_1$ for all $t \in [0, T_1)$. Due to the choice of D and the first estimate in (40) we have

$$R_1 \geqslant R = D \cdot \varepsilon^{-(\alpha - \dim \Omega/4) + \alpha/\sigma + \rho} \cdot ||U_0 - (\bar{u}_0, \bar{v}_0)||_*^{\rho} \geqslant ||U_0 - (\bar{u}_0, \bar{v}_0)||_*.$$

Finally, let $T_0 \in [0, T_1]$ denote the maximal time such that $||U(t) - (\bar{u}_0, \bar{v}_0)||_* < R$ for all $t \in [0, T_0)$. Then according to [24, Theorem 2.10], (39) holds for all $t \in [0, T_0]$, and in order to finish the proof we only have to verify $||U(T_0) - (\bar{u}_0, \bar{v}_0)||_* = R$.

Since this is satisfied if $T_0 < T_{\max}$, assume $T_0 = T_{\max}$ and $||U(T_0) - (\bar{u}_0, \bar{v}_0)||_* < R$. We have already seen that $R < R_\varepsilon$, so the definition of $T_{\max}$ shows that $U(T_0)$ has to lie on the boundary of the cone $(\bar{u}_0, \bar{v}_0) + \mathscr{K}_{2\delta_\varepsilon}$. On the other hand, estimate (39) is satisfied for all $t \in [0, T_{\max}]$. Since the cone $\mathscr{K}_{\delta_\varepsilon}$ is positively invariant for linear solutions, $U_{\text{lin}}(T_0) \in \mathscr{K}_{\delta_\varepsilon}$. Lemma 2.8 in [24] shows that $U(T_0) \in (\bar{u}_0, \bar{v}_0) + \mathscr{K}_{c \cdot \delta_\varepsilon}$ for some $c < 2$, i.e., it cannot be contained in the boundary of $(\bar{u}_0, \bar{v}_0) + \mathscr{K}_{2\delta_\varepsilon}$. Thus we have a contradiction, and $T_0 < T_{\max}$. This completes the proof. □

7. Conclusions

We end with a few observations based on our simulations for the Thomas equation. This paper shows that the initial pattern selection in certain systems with Turing instabilities is determined by the linearized equation. This initial behavior is important for understanding longer-time behavior. In one dimension, we have observed numerically that solutions converge to an equilibrium with the same qualitative features as the initial selection. That is, for a fixed set of parameters there can be many coexisting stable equilibria; different initial conditions lead to convergence to different equilibria. It is the initial pattern selection described here which determines which of these equilibria is ultimately observed. For two-dimensional domains, our simulations indicate that the initial linear pattern selection gives qualitative information for the solutions for a long time. As in one dimension, as time goes to infinity, solutions will converge to an equilibrium. However, we suspect that these equilibria are not what is observed in animal coats. Since the pattern selection for animal coats is a finite time process, the "large" time rather than infinite time behavior is relevant for the pattern selection. We intend to address this issue further in the future.

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