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Period-doubling cascades for large perturbations of Hénon families

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Abstract. The Hénon family has been shown to have period-doubling cascades. We show here that the same occurs for a much larger class: Large perturbations do not destroy cascades. Furthermore, we can classify the period of a cascade in terms of the set of orbits it contains, and count the number of cascades of each period. This class of families extends a general theory explaining why cascades occur [5].

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1. Introduction

One of the most mysterious phenomena in nonlinear dynamics is period-doubling cascades. Cascades never occur alone. Processes have infinitely many cascades if they have one, and they are seen in a wide variety of numerical and experimental investigations. As a rule of thumb, for systems that depend on a parameter, it seems that as systems become more chaotic, we see period-doubling cascades. We have taken a more austere view, that if a system is nonchaotic for very negative values of its parameter and is fully chaotic in some sense for very large positive parameter values, then for parameter values in between there are period-doubling cascades, independent of how complicated the transition is from no chaos to full chaos. By *the period* of a periodic orbit, we mean its least period.

A cascade is a special type of connected set of periodic orbits, connected in the space of periodic orbits under the Hausdorff metric. As a first approximation to a definition, we say a one-parameter family of maps $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ has a *period-k cascade* if there is a (connected) path in $\mathbb{R} \times \mathbb{R}^n$ of periodic points such that the set of their periods is $\{k, 2k, 4k, 8k, \dots\}$ (not necessarily occurring in this order, and listed without multiplicity). We restrict these paths to certain "non-flip" periodic orbits. The set of periodic orbits can be a collection of complicated networks of periodic orbits, and this restriction prunes the network to manageable simplicity. Specifically, flip orbits are those whose Jacobian matrix has an odd number of real eigenvalues that are less than -1 and has none which are equal to -1. Nonflip orbits are all the rest. Later, we restate this more formally. If the cascade can be chosen so that its closure in $\mathbb{R} \times \mathbb{R}^n$ is not compact, then we call the cascade *unbounded*. We have developed a theory which explains (and counts) the occurrence of cascades under general conditions for generic one-parameter families of n-dimensional maps for arbitrary n. In this paper, we show that a family obtained by adding a generic arbitrarily large perturbation to the Hénon family retains the same cascade structure as the unperturbed family.

The Hénon family

$$H_A(x, y) = \begin{pmatrix} A + By - x^2 \\ x \end{pmatrix}$$

is a much studied dynamical example. In an early result in the field, Devaney and Nitecki [1] showed that for any fixed nonzero B, as A varies from small to large, the Hénon family forms a horseshoe. Specifically, all the interesting dynamics is captured by looking at a certain rectangular region of the plane: For sufficiently negative A there are no bounded trajectories, but for large positive A the invariant set in this region has dynamics of a Smale horseshoe. In particular, the invariant set is a hyperbolic set with one expanding and one contracting direction, and the dynamics on the set is topologically conjugate to the full shift on two symbols. Both [2] and [6] showed that as the Hénon horseshoe forms, the family has infinitely many cascades. We now show that this result holds for a broader class of families. Some of the difficulties are hinted at in [3] which shows that the familiar monotonicity of orbit creation in the one-dimensional quadratic map is essentially never true for chaotic families of diffeomorphisms in the plane. Orbits are destroyed as well as created as its parameter increases.

We now define the perturbations of the Hénon family that we will consider. Note that the word perturbation is usually used to denote something small. In this paper the perturbations can be artitrarily large, as long they are small in comparison to the original Hénon family in the asymptotic limit as x, y, and A go to infinity. We will investigate maps $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$F(A, x, y) = \begin{pmatrix} A + By - x^2 + g(A, x) + \alpha_1(A, x, y) \\ x + \alpha_2(A, x, y) \end{pmatrix}. \tag{1}$$

In each case, B is a fixed nonzero constant, and A is the bifurcation parameter. The functions g and $\alpha = (\alpha_1, \alpha_2)$ satisfy conditions stated below.

The class of functions g permitted are given as follows: Fix $\beta > 0$. Define G_β for $\beta > 0$ to be the set of C^∞ functions $g : \mathbb{R}^2 \to \mathbb{R}$ such that for all $(A, x) \in \mathbb{R}^2$,

$$|g(A, 0)| < \beta \quad \text{and} \quad |\partial g / \partial x(A, x)| < \beta.$$

This class includes for example C^∞ functions that are C^1 bounded.

We now describe the class of functions allowed for α. Fix $r > 0$ to be any arbitrarily large constant. For sufficiently small $\delta > 0$ depending on r, let

$$\Psi_{\{\delta,r\}} = \{\alpha : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \in C^\infty : \|\alpha(A,x,y)\|_1 < \delta \text{ when } \|(A,x,y)\| > r\}$$

(where $\|\cdot\|_1$ denotes the C^1-norm). Notice that this class of perturbations has no restrictions other than smoothness in the region where $\|(A,x,y)\| < r$. This class includes for example all C^∞ functions with compact support, since any C^∞ function with compact support is contained in $\Psi_{\{\delta,r\}}$ for some r. Why do we not just assume $\alpha \equiv 0$ outside a ball or radius r? Allowing α to be nonzero everywhere means that that the set of allowable functions F is open in C^∞. Thus there exists a residual subset of this open set (the set depends on g) in which all periodic orbit bifurcations are generic. (See Definition 5.) Generic bifurcations allow us to describe the connected sets of periodic orbits, which is essential for our task.

Since the function α is uniformly bounded, we can find a constant β_1 so that for all $(A,x,y) \in \mathbb{R}^3$,

$$|g(A,0)| + |\alpha(A,x,y)| < \beta_1 \quad \text{and} \quad |\partial g/\partial x(A,x)| < \beta_1.$$

In a slight abuse of notation, we will drop the subscript 1, and just refer to this new larger constant as β.

We have chosen these perturbations so that they are dominated by the standard Hénon terms when A and x are large. In particular, for sufficiently large $A = A_1$ and sufficiently small $\delta > 0$, the map $F(A_1,\cdot,\cdot)$ is topologically the same as the Hénon map. We show that for A equal to any sufficiently large A_1, any nonflip orbit in the horseshoe will lie in a cascade, and no other orbit for that A_1 is in the cascade.

Definition 1. Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ function. We write $[p]$ for the orbit of the periodic point p. If p is a periodic point for $f(A,\cdot)$, then in a slight abuse of terminology, we say that $\sigma = (A,p)$ is a periodic point for F. We write $[\sigma]$ or $(A,[p])$ for the orbit. We denote the set of periodic orbits in $\mathbb{R} \times \mathbb{R}^n$ under the *Hausdorff metric* by $\mathrm{PO}(f)$.

Let $\sigma = (A,p)$ be a periodic point of period k of a smooth map $G = f(A,\cdot)$. We refer to the *eigenvalues of* σ *or* $[\sigma]$ as shorthand for the eigenvalues of the Jacobian matrix $DG^k(p)$. Of course all the points of an orbit have the same eigenvalues. We say that $[\sigma]$ is *hyperbolic* if none of its eigenvalues have norm 1.

Let $[\sigma]$ be a period-k orbit for f. We call $[\sigma]$ a *nonflip* orbit for f if $[\sigma]$ has an odd number of real eigenvalues less than -1 and no eigenvalues $= -1$.

We denote the set of all nonflip orbits in $\mathbb{R} \times \mathbb{R}^n$ under the Hausdorff metric by $\mathrm{PO}_{\mathrm{nonflip}}(f)$.

Definition 2. An *open arc* is a set which is homeomorphic to an open interval.

Definition 3. A *(period-doubling) cascade of period* m is an open arc in $\mathrm{PO}_{\mathrm{nonflip}}(f)$ with the following properties:

(i) The open arc contains orbits of period $2^k m$ for some positive integer m and for every nonnegative integer k.

(ii) The number m is the smallest integer for which this is true, and m cannot be made smaller by making the open arc larger.

Let (p_k) be the sequence of periods of the nonhyperbolic orbits, ordered so that for each k, the $k+1$ orbit lies along the open arc between the k orbit and the $k+2$ orbit. Under our genericity hypotheses, it turns out that no period can occur in the sequence infinitely many times. It follows that in at least one direction ($k \to \infty$ or $k \to -\infty$), $\lim_k p_k = \infty$.

We say the cascade is *unbounded* if it does not lie in a compact set of $\mathbb{R} \times \mathbb{R}^n$.

Even orbits for the two-shift. For any fixed k, consider a period-k orbit $\overline{S}$ of the full shift on two symbols. This orbit is associated with a length-k sequence of two symbols: $S = (a_1, \ldots, a_k)$, where each a_k is equal to either the symbol -1 or $+1$, and S is not periodic. We say that $\overline{S}$ is *even* if the associated finite sequence S has an even number of -1's (or more compactly, if $\prod_{j=1}^{k} a_k = 1$).

For a map $F(A, \cdot, \cdot)$ of the form in (1), define $\text{MaxInv}(A)$ to be the union of the trajectories such that all positive and negative iterates are bounded.

Our main theorem is as follows:

Theorem 4 (Cascades for large perturbations of Hénon families). *Fix $B \neq 0$, $\beta > 0$, and $r > 0$. Let $g \in G_\beta$. For $\delta > 0$, let $\alpha \in \Psi_{\{\delta,r\}}$, and let F be as in* (1)*. Then as long as δ is sufficiently small (depending on r), for every sufficiently large $A = A_1$ depending on β, r, and B, there is a residual set of $\alpha \in \Psi_{\{\delta,r\}}$ depending on the function g and the constant B for which the following hold:*

1. $\text{MaxInv}(A_1)$ *is conjugate under a homeomorphism to a two-shift, and this homeomorphism gives a one-to-one correspondence of the even symbol sequences with the nonflip orbits. (Hence for A_1, we can without confusion refer to a periodic orbit for $F(A_1, \cdot)$ as being even.)*
2. *Each unbounded cascade contains exactly one periodic orbit for $F(A_1, \cdot)$, and it is an even orbit.*
3. *For each even orbit there is a unique unbounded cascade containing that orbit.*
4. *If an even periodic orbit is of period k, and k is odd, the cascade containing it is a period-k cascade. If k is even, then the cascade containing it is a period-j cascade, where $k/j = 2^m$ for some m.*

In other words, corresponding to every even period-k symbol sequence S, there is exactly one unbounded cascade of F that satisfies the following: At parameter value A_1, the cascade contains a unique periodic orbit, and it is the unique period-k orbit of F with the symbol sequence S.

The number of even period-k orbits. Note that the number of period-k points of the two-shift for each k has been studied quite extensively, and is often referred to as the ζ function. In some cases, it is possible to write an easy formula for the number $\Gamma(2, k)$ of *even* period-k orbits, as follows: There is one even fixed point, and no even period-2 orbits. If the k is an odd prime, the number of even period-k

orbits is $(2^k - 2)/(2k)$. In general, if k is odd, the number of even period-k orbits is exactly half the number of period-k orbits.

In the general case, any positive integer k, let $L(k) = \Sigma(\Gamma(2, j))$ for all $j < k$ for which k/j is a power of 2. Of course $L(k) = 0$ if k is odd. Then

$$\Gamma(2, k) = (\zeta(2, k)/k - L(k))/2.$$

See [5] for a detailed discussion.

2. Proof of Theorem 4

Let B, F, α and g be as in the statement of the theorem. Note that the assumptions on g and α imply that for all $(A, x, y) \in \mathbb{R}^3$,

$$|g(A, x)| + |\alpha(A, x, y)| < \beta(1 + |x|).$$

Define $s = s(A_1) = \sqrt{A_1}$, and let $Q = 2s$. Let the square E be defined by $E = [-Q, Q] \times [-Q, Q]$. Assume $A_1 > r$ and $A_1 > Q$. Additional lower bounds will be placed on A_1.

The proof of the theorem proceeds from the following steps:

Step 1: Horseshoe dynamics for large A_1. Set $\alpha(A, x, y) \equiv 0$. For A_1 sufficiently large and for $\delta > 0$ sufficiently small, the following are true:

(1a) *Periodic orbits in E.* For all $A < A_1$, all periodic orbits of F are contained in the interior of E.

(1b) *The two-shift.* On $\mathrm{MaxInv}(A_1)$, F is topologically conjugate to the full shift on two symbols.

(1c) *Hyperbolicity.* F is hyperbolic on $\mathrm{MaxInv}(A_1)$, with one expanding and one contracting direction at each point.

(1d) *Nonflip orbits.* For $A = A_1$, the nonflip period-k orbits of F are in one-to-one correspondence with the even period-k orbits of the full shift on two symbols.

Step 2: Adding small perturbations. The results in Step 1 are not sensitive to C^1-small perturbations. Thus, they are still true when we add $\alpha(A, x, y) \in \Psi_{\{\delta,r\}}$ for sufficiently small $\delta > 0$, since we assume that $A_1 > r$, implying that $\|\alpha\|_1 < \delta$ for $A = A_1$.

Step 3: No orbits for small A_0. For fixed A_0 sufficiently negative (and in particular $A_0 < -r$), the map F has no periodic orbits.

Step 4: Cascades. Let $\alpha(A, x, y)$ be contained in a residual set of $\Psi_{\{\delta,r\}}$ such that all bifurcations of F are generic (generic bifurcations are defined below). Each nonflip periodic orbit of $F(A_1, \cdot, \cdot)$ is contained in a unique unbounded cascade.

Step 5: Period of the cascades. If k is an odd number, this unbounded cascade is a period-k cascade of F. If k is an even number, then this unbounded cascade is a period-j cascade of F, where the ratio k/j is a power of two.

Proof of Step 1: Horseshoe dynamics for large A_1.

(1a) *Periodic orbits in* E and (1b) *The two-shift.* Let F be of the form in (1) with $\alpha \equiv 0$. Let $L = [-Q, Q]$. Let $J_1 = [-2s, -s/2]$, $J_2 = [s/2, 2s]$, $J = J_1 \cup J_2$.

We have previously shown in [5] that for all sufficiently large parameter values λ_1, the quadratic map $q(\lambda, x) = \lambda - x^2 + h(\lambda, x)$, where $|h| < \beta(1 + |x|)$, has the following properties:

Q1. $q(\lambda_1, L \setminus J)$ contains no points of L.

Q2. There is an interval M in L such that for all $\lambda \leq \lambda_1$, each periodic orbit is contained in M.

Q3. At λ_1, $q(\lambda_1, J_i)$ maps diffeomorphically across L, where $i = 1$ or 2.

For sufficiently large A_1, we get similar results for $F = (F_1, F_2)$:

F1. For sufficiently large A_1, $F(A_1, E \setminus \{J \times L\})$ contains no points of E. This is an immediate consequence of Q1 above, since $F_1(A_1, x, y) < A_1 + |B|Q - x^2$ inside E.

F2. For all $A < A_1$, all periodic orbits are contained in the interior of E. This is not immediate from the quadratic case. The proof is as follows. Let $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ be a periodic orbit at parameter A. Fix x to be the x_i with the maximum absolute value. Let y be the corresponding y_i. Thus $y_i = x_{i-1}$. Let $\overline{x} = x_{i+1}$. That is, $F_1(A, x, y) = \overline{x}$. Thus $|y| < |x|$ and $|\overline{x}| < |x|$. This implies that
$$-|x| < F_1(A, x, y) \leq A - x^2 + |B|\,|y| + \beta(1 + |x|).$$
Since $|y| < |x|$, we have $0 \leq (A + \beta) - x^2 + |x|(|B| + \beta + 1)$. Let $\rho = (|B| + \beta + 1)/2$. Then
$$0 \leq (A + \beta) + \rho^2 - (|x| - \rho)^2.$$
Hence
$$|x| \leq \rho + \sqrt{A + \beta + \rho^2}.$$
Note that this right-hand side is increasing in A. Since B, β, and thus ρ are fixed, for A_1 sufficiently large,
$$\rho + \sqrt{A_1 + \beta + \rho^2} < 2\sqrt{A_1} = Q.$$
Thus as long as $A \leq A_1$, we have $|x| < Q$. Since x is the point of the orbit with the maximum absolute value, this implies that the periodic orbit is contained in the interior of E.

F3. At $A = A_1$, for each fixed y, $F_1(A_1, J_i \times \{y\})$ maps diffeomorphically across L. This is immediate from Q3 above. In addition, $F_2(A_1, J_i \times L) = J_i$. Therefore, each $J_i \times L$ maps diffeomorphically, across E, and vertically staying inside of E, such that the $i = 1, 2$ images are disjoint.

(1c) *Hyperbolicity.* Assume

$$\sqrt{A_1} > \beta + |B| + \max\{1, |B|\}.$$

The determinant and trace of the Jacobian matrix of F are respectively $-B$ and $-2x + \partial g/\partial x$. For any point in $J \times L$,

$$|-2x + \partial g/\partial x| > 2s - \beta = 2\sqrt{A_1} - \beta.$$

The assumption above on A_1 implies that one eigenvalue for the Jacobian matrix is contracting and the other expanding.

Define the *stable* and *unstable cones* respectively by

$$S_c^+ = \{(\xi, \eta) : |\xi| \geq c|\eta|\}, \qquad S_c^- = \{(\xi, \eta) : |\xi| \leq c|\eta|\}.$$

Then for $A = A_1$ and any point in $J \times L$, the Jacobian matrix DF maps S_1^+ into S_N^+, where $N = \sqrt{A_1} - \beta - |B|$, and DF^{-1} maps S_1^- into $S_{N_1}^-$, where $N_1 = N/|B|$. To see this, let $(\xi, \eta) \in S_1^+$, and let $(\xi_1, \eta_1) = DF(\xi, \eta)$. Then

$$(\xi_1, \eta_1) = ((-2x + g_x)\xi + B\eta, \xi).$$

Thus

$$\frac{|\xi_1|}{|\eta_1|} \geq \frac{(\sqrt{A_1} - \beta)|\xi| - |B|\,|\eta|}{|\xi|} \geq \sqrt{A_1} - \beta - |B| = N > 1.$$

Therefore DF maps S_1^+ into the interior of itself. Likewise, let $(\xi, \eta) \in S_1^-$, and let $(\xi_{-1}, \eta_{-1}) = DF^{-1}(\xi, \eta)$. Then

$$\begin{aligned} \frac{|\eta_{-1}|}{|\xi_{-1}|} &= \frac{1}{|B|} \frac{|B\xi + (2x - g_x)\eta|}{|\eta|} \geq \frac{(\sqrt{A_1} - \beta)|\eta| - |B|\,|\xi|}{|B|\,|\xi|} \\ &\geq \frac{\sqrt{A_1} - \beta - |B|}{|B|} = N_1 > 1. \end{aligned}$$

Therefore DF^{-1} maps S_1^- into the interior of itself.

Thus the stable and unstable cones are mapped strictly inside themselves and expanded respectively under the derivative and its inverse. Using the method of cones (Corollary 6.4.8 in [4]), this guarantees that at $A = A_1$, F is hyperbolic on MaxInv(F).

Putting this together, we find that at A_1, F on MaxInv(F) is hyperbolic and topologically conjugate to the two-shift. Specifically, we know that MaxInv(F) is contained in $\{J_1 \cup J_2\} \times L$. The conjugacy codes a point by considering its bi-infinite orbit. For any integer i, we code the i^{th} point in the itinerary of an orbit with a "1" if the i^{th} iterate is in the left region $J_1 \times L$, and with a "-1" if the i^{th} iterate is in the right region $J_2 \times L$.

(1d) *Nonflip orbits.* We now determine the nonflip orbits for F for $A = A_1$. In the left region $J_1 \times L$, DF has an expanding eigenvalue which is greater than 1, whereas for the right region $J_2 \times L$, DF has an expanding derivative which is less than -1. Thus any period-k orbit $[p]$ in $J \times L$ is a nonflip orbit exactly when $[p]$ is in $J_2 \times L$ an even number of times. Thus by our conjugacy, there is the one-to-one correspondence between nonflip orbits of F on $\mathrm{MaxInv}(F)$ and the even orbits for the two-shift.

Proof of Step 2: Adding small perturbations. For large $|A_1|$, we have established hyperbolic dynamics on an invariant horseshoe in $\mathrm{MaxInv}(F)$. All of this is robust under sufficiently C^1-small additive perturbations $\alpha(A_1, x, y)$, since C^1-small implies that we can make both the function values and all the partial derivatives as small as we want. Furthermore, as long as $\alpha(A, x, y) \in \Psi_{\{\delta,r\}}$ for any $A \in [A_0, A_1]$, α is C^1-small for $|(x, y)| > r > Q$. Therefore $\mathrm{MaxInv}(F) \subset E$ for all $A \in [A_0, A_1]$.

Proof of Step 3: No orbits for small A_0. From F2, it suffices to show that for sufficiently negative A_0, $F(A_0, E) \cap E$ is empty. Note that for all $(x, y) \in E$,

$$\begin{aligned} F_1(A_0, x, y) &= A_0 + By - x^2 + g(A_0, x, y) + \alpha_1(A_0, x, y) \\ &< A_0 + \beta + |B|Q + \beta|x| - x^2. \end{aligned}$$

This quadratic in $|x|$ has a maximum at $|x| = \beta/2$, implying that

$$F_1(A_0, x, y) < A_0 + \beta + |B|Q + \beta^2/4.$$

Thus as long as $A_0 + \beta + (|B| + 1)Q + \beta^2/4 < 0$, $F_1(A_0, x, y) < -Q$ for any $(x, y) \in E$, implying that $F(A_0, x, y)$ is not contained in E.

Proof of Step 4: Cascades. In order to prove our theorem, we state the following abstract results on the existence of cascades, from [5]:

Definition 5. Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be C^∞. Let U be an open subset of $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$, and let V be its closure. By *periodic orbit bifurcation*, we mean a change (as a parameter is varied) in the local number of periodic orbits or a change in the dimension of their unstable space. We refer to a periodic orbit bifurcation in U as *generic* if it is one of the following three types:

- A generic saddle-node bifurcation.
- A generic period-doubling bifurcation.
- A generic Hopf bifurcation with no eigenvalues which are roots of unity.

In [5], we show that a residual set of one-parameter families have only generic periodic orbit bifurcations. Let $\alpha(A, x, y) \in \Psi_{\{\delta,r\}}$ be such that our F has only generic bifurcations.

We now define the periodic orbit index in a way that is specific to $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$. Let P be a hyperbolic period-k orbit for F with the eigenvalues $\sigma_1 \le \sigma_2$ for derivative $D(F^k)$ with respect to the spatial variables (x, y). Let $I_m = (-\infty, -1)$, $I_0 = (-1, 1)$, and $I_p = (1, \infty)$. We define the *periodic orbit index* for P to be

$$\mathrm{ind}_F(P) = \begin{cases} 1 & \text{if } \sigma_1, \sigma_2 \text{ are both in } I_m,\ I_0, \text{ or } I_p, \text{ or if } \sigma_1, \sigma_2 \text{ are complex}, \\ -1 & \text{if } \sigma_1 \in I_0 \text{ and } \sigma_2 \in I_p, \\ 0 & \text{if } \sigma_1 \in I_m \text{ and } \sigma_2 \notin I_m. \end{cases}$$

Note that a flip orbit corresponds to the case of index 0. For large parameters such as A_1, all periodic orbits are saddles, implying that the periodic orbit index is -1 or 0.

This definition generalizes to a general definition for $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. It is a topological invariant, as is described in more generality and detail in [5]. Let $[\Gamma] : (0,1) \to \mathrm{PO}_{\mathrm{nonflip}}(f)$ map homeomorphically to an open arc C in the nonflip orbits for f. (The brackets are to emphasize that $[\Gamma]$ maps a point in the interval $(0,1)$ to an orbit in the set of nonflip orbits.) Then $[\Gamma]$ can be identified with one of the two orientations on C. There is one orientation that is induced by the periodic orbit index: $[\Gamma]$ is an *index orientation* on C as long as it has the following property for every $s \in (0,1)$: $\mathrm{ind}_f([\Gamma(s)]) = -1$ whenever the parameter A is locally decreasing, and $\mathrm{ind}_f([\Gamma(s)]) = +1$ whenever the parameter A is locally increasing. As long as f has only generic bifurcations, there exists an index orientation on every open arc in the set of nonflip periodic orbits.

Hypothesis 6 (Orbits near the boundary). Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ function. Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be an open set such that f has only generic bifurcations in U. Assume also that the set of periodic points of f in U is contained in a bounded set in $\mathbb{R}^{n+1}$. Let J be an interval, and let Q be an open arc in $\mathrm{PO}_{\mathrm{nonflip}}(f)$ with index orientation $\sigma : J \to Q$ such that that Q is contained in U, though the image of its endpoints may not be. If $[\sigma(\inf J)]$ is defined and contained in $V \setminus U$, then it is called an *entry orbit* of U. Denote the set of all entry orbits of U by IN. If $[\sigma(\sup J)]$ is defined and is contained in $V \setminus U$, then it is called an *exit orbit* of U. Denote the set of all exit orbits of U by OUT. Assume that $\mathrm{IN} \cap \mathrm{OUT}$ is empty.

In our case, this hypothesis is satisfied for F for the set $U = \{[A_0, A_1] \times E\}$. In fact, the set of exit orbits for U is empty, and the set IN of nonflip orbits on the boundary of the region U is entirely contained in $\{A_1 \times \mathrm{int}(E)\}$.

Theorem 7 (General cascades theorem). *Assume Hypothesis* 6*. For any odd number d, consider the set of orbits in* $\mathrm{PO}_{\mathrm{nonflip}}(f)$ *with period $2^k d$ for a positive integer k. Let* IN_d *be the set of entry periodic orbits of this type in* $\mathrm{PO}_{\mathrm{nonflip}}(f)$*. Let* OUT_d *be the set of exit periodic orbits of this type in* $\mathrm{PO}_{\mathrm{nonflip}}(f)$*. Assume that* IN_d *contains K elements, and* OUT_d *contains J elements. We allow one but not both of J and K to be infinite. If $K < J$, then all but K members of* OUT_d *are contained in distinct period-doubling cascades. Likewise, if $J < K$, then all but J members of* IN_d *are contained in distinct period-doubling cascades.*

This theorem is proved in [5]. From this abstract theorem, we conclude that each nonflip periodic orbit P for our $F(A_1, \cdot, \cdot)$ is contained in a unique cascade. We have already shown that there is a unique nonflip periodic orbit for $F(A_1, \cdot, \cdot)$ corresponding to each even orbit for the two-shift.

Proof of Step 5: Period of the cascades. The only type of generic bifurcations which change the period of an orbit are period-doubling and period-halving bifurcations. If the period k of P is odd, then the period is already minimal in the sense that it is not possible to bifurcate to half the period. Therefore the period of the cascade through P is k. If the period k is even, then it is possible that there is a period-halving bifurcation within the cascade, implying that the period of the cascade is less than the period k of the orbit P. However, the period always changes by a factor of two, implying that the ratio of the period of P and the period of the cascade is a power of two.

This completes the proof of the theorem. □

2.1. Reduced smoothness conjecture

We end with a conjecture about the nongeneric version of our abstract theorem. If proved, in the context of the current paper, this would imply that it is possible to extend the results on perturbed Hénon families to the case of nongeneric perturbations. The barrier to proving Theorem 7 in the nongeneric case is that it is not possible to control the behavior of eigenvalues other than those involved in the bifurcations. Such control is needed for the limiting arguments to work. Otherwise, a sequence of maps with periodic orbits of fixed period can limit to a map with a periodic orbit of smaller period. Lefschetz number arguments, as used by Franks [2], do not apply to Theorem 7, since the hypotheses use the orbit index, and thus do not include any assumptions on the Morse index of the flip orbits. The following is a generalization of the definition of orientation.

Definition 8. Assume that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous family such that there are only hyperbolic periodic orbits on the boundary of a region U in $\mathbb{R} \times \mathbb{R}^n$. We call an orbit $[q]$ a *generalized entry orbit* if the Morse index of q is even, and a *generalized exit orbit* if the Morse index is odd.

Every orbit which is an entry (resp. exit) orbit is a generalized entry (resp. exit) orbit, but the generalized orientation is also defined for flip orbits.

Conjecture 9 (Abstract result reformulated). *Assume that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous family and that $U \subset R \times \mathbb{R}^n$ is such that the periodic orbits in U are contained in a bounded region. Assume also that on the boundary of U, there are only hyperbolic periodic orbits. Let IN_d and OUT_d respectively be the generalized entry and exit orbits on the boundary of U of period $p = 2^m d$, where d is a fixed odd number, and m is any positive integer. Assume that the number of orbits in OUT_d and the number of orbits in IN_d differ. Let K be the smaller of the two. (We allow one but not both of these numbers to be infinite.) Then there is a cascade through all but possibly K of the orbits in the larger of the sets IN_d and OUT_d.*

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