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A period-doubling cascade precedes chaos for planar maps

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A period-doubling cascade is often seen in numerical studies of those smooth (one-parameter families of) maps for which as the parameter is varied, the map transitions from one without chaos to one with chaos. Our emphasis in this paper is on establishing the existence of such a cascade for many maps with phase space dimension 2. We use continuation methods to show the following: under certain general assumptions, if at one parameter there are only finitely many periodic orbits, and at another parameter value there is chaos, then between those two parameter values there must be a cascade. We investigate only families that are generic in the sense that all periodic orbit bifurcations are generic. Our method of proof in showing there is one cascade is to show there must be infinitely many cascades. We discuss in detail two-dimensional families like those which arise as a time-2π maps for the Duffing equation and the forced damped pendulum equation. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4813600]

In a series of papers in 1958–1963, Pekka Myrberg was the first to discover that as a parameter is varied in the one-dimensional quadratic map (cf. Figure 1), periodic orbits of periods *k*, 2*k*, 4*k*, 8*k*, … occur for a variety of *k* values (cf. Refs. 45 and 46 and references therein). This now well known phenomenon of period-doubling cascades has been seen in a large variety of parameter-dependent dynamical systems. Though it is not always possible to detect cascades numerically, as they may not be stable, they have been observed numerically in the contexts of maps, ordinary differential equations, partial differential equations, delay differential equations, and even in physical experiments. See Refs. 6–9, 21, 24, 26–28, 31, 38, 41, 54, 55, 60, and 62 and the double-well Duffing equation in Figure 2. It has often been observed that cascades occur in a dynamical system varying with a parameter during the transition from a system without chaos to one with chaos. In the current paper, we rigorously show the existence of a cascade (in fact of infinitely many) for certain two-dimensional systems, which include a transition to chaos.

I. INTRODUCTION

Though period-doubling cascades (subsequently shortened to *cascades*) frequently occur en route to chaos, the onset of chaos can occur without cascades. For example, chaos appears following the appearance of a homoclinic orbit (without a cascade) in the Lorenz system as ρ is increased past $\rho \sim 13.9$ (for fixed $\beta = 8/3$ and $\sigma = 10$).[1,34] The Lorenz system has cascades but only for larger ρ values.[56] A second example is the parameterized tent map family $f(\lambda, x) = 1 - \lambda|x|$ on $[-1,1]$; for $\lambda<1$ there is no chaos, and for all $\lambda>1$ there is a set on which chaos occurs. However, there are no cascades for the tent map family. We note here that in using the term "chaos," we include transient chaos; we do not assume that there is a chaotic set that is an attractor. These examples suggest that it is of value to understand better when cascades must accompany the transition to chaos. Observe that the tent map is not differentiable, and the Lorenz system cannot be written as a map defined on two-dimensional manifold. (Standard representations as a Poincaré map result in places where the map is not defined. While one can study a Poincaré return map using a plane defined by having a constant z, the return map is not defined at those points in the plane that are on the stable manifold of the origin and their trajectories never return to the plane.) In this paper, we restrict to smooth (differentiable) maps meaning that the tent map, its two-dimensional analogues, and the Lorenz system do not fit within our theory. In particular, our focus in this paper is on one-parameter families of two-dimensional maps, which throughout this paper have the form

$$F(\lambda, x), \quad \text{with} \quad \lambda \in \mathbb{R} \quad \text{and} \quad x \in \mathfrak{M}, \tag{1}$$

where $\mathfrak{M}$ is a 2-dimensional manifold such as $\mathbb{R}^2$ or a cylinder. Such a map arises from the time-2π map for double-well Duffing equation (cf. Figure 2).

Types of orbits. A (periodic) orbit for a map F is a finite set of points. That is, for a fixed λ value and finite k, a period-k orbit of $F(\lambda, \cdot)$ is a finite set of points $\{x_1, \ldots, x_k\}$ such that $F(\lambda, x_i) = x_{i+1}$ for $i < k$ and $F(\lambda, x_k) = x_1$. In what follows we refer to a periodic orbit as an orbit and call it a saddle if its Jacobian matrix $D_x F^k(\lambda, x)$ has eigenvalues α and β satisfying $|\alpha| < 1 < |\beta|$. The saddle is a regular (or nonflip) saddle if $\beta > 1$ and is a flip saddle if $\beta < -1$. A regular orbit is any orbit that is not a flip saddle. If the orbit has an eigenvalue with absolute value 1, it is called a bifurcation orbit. An orbit that is not a bifurcation orbit is a flip orbit if $D_X F$

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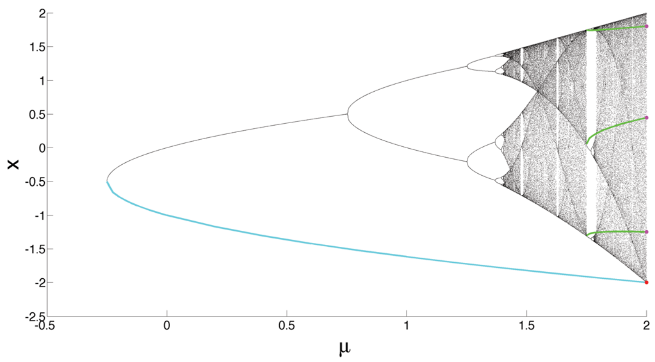

FIG. 1. *The parameterized map* $F(\mu, x) = \mu - x^2$. For each fixed μ value, the attracting set in [0, 1] is shown. There are infinitely many cascades. For this map, each cascade has precisely one saddle-node bifurcation. The branch of unstable orbits ending at the saddle-node bifurcation is shown for period-1 and period-3. Each regular periodic orbit within the map's horseshoe at $\mu = 2$ is uniquely connected to one of the (infinitely many) cascades. The regular period-one and period-three orbits are shown here, depicted respectively by one and three dots at $\mu = 2$. This is the bifurcation diagram most frequently displayed to illustrate the phenomenon of period-doubling cascades. However, cascades occur for much more complex dynamical systems that are completely unrelated to quadratic maps, as shown for example in Fig. 2.

has precisely one real eigenvalue that is less than -1. Note that a flip orbit is either a saddle or a repeller.

If a cascade exists then for some k, there is a sequence of parameter values $\lambda_0, \lambda_1, \lambda_2, \ldots$ such that there is a period-doubling bifurcation of a period-k orbit for $F(\lambda_0, \cdot)$, a period-doubling bifurcation from a period-$2k$ orbit for $F(\lambda_1, \cdot)$, a period-doubling bifurcation of a period-$4k$ orbit for $F(\lambda_2, \cdot)$, and in general for every $i > 0$, there is a period-doubling bifurcation of a period-$2^i k$ orbit for $F(\lambda_i, \cdot)$.

There are many definitions of chaos, the choice of which depends on the aspect of chaos to be emphasized. The definition that is appropriate here is based on periodic orbits. We say F has periodic-orbit (PO) chaos or is PO-chaotic at λ_1 if $F(\lambda_1, \cdot)$ has infinitely many regular periodic saddles. For example, if a map has a horseshoe, it must have PO-chaos.

We consider only maps F in (1) which are generic in the sense that all its bifurcation orbits are generic; specifically, we say a bifurcation orbit P of F is generic if it is one of the following three types (cf. Ref. 50):

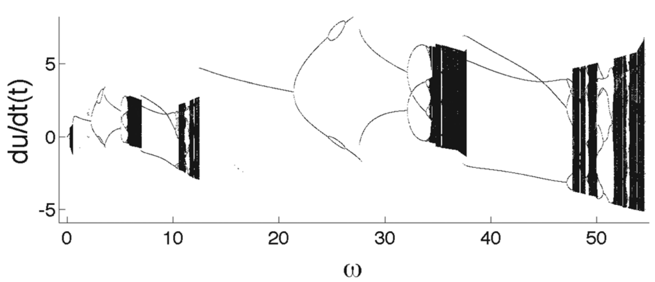

FIG. 2. *The attracting set for the double-well Duffing equation:* $u''(t) + 0.3u'(t) - u(t) + (u(t))^3 + 0.01 = \omega \sin t$. This equation is periodically forced with period 2π. Therefore, the $F =$ time-2π map is a diffeomorphism on $\mathbb{R}^2$ parameterized by ω. Depicted here is the attracting set of F, projected to the $(\omega, u'(t))$-plane. The constant 0.01 has been added to destroy symmetry in order to avoid non-generic symmetry-breaking bifurcations.

(i) A generic saddle-node bifurcation.
(ii) A generic period-doubling bifurcation.
(iii) A generic Hopf bifurcation for a map, i.e., a Naimark-Sacker bifurcation.

See Ref. 52 for a more detailed description of generic bifurcations. We say F is generic when each of its bifurcation orbits is generic. For example, no period-tripling bifurcations or other degenerate bifurcations are permitted. In particular, such maps F are dense in set of smooth families in a space C^∞ of infinitely differentiable functions. See Ref. 52 for a more detailed technical description of this generic set of maps.

Main result. The following says that in the class of F we consider, there is a cascade if the system becomes PO-chaotic.

Theorem 1 (Route to PO-chaos). *Assume that F in (1) is generic (as described above), and that*

(S_0) *$F_0 = F(\lambda_0, \cdot)$ has at most a finite number of orbits, and it has no bifurcation orbits.*
(S_1) *$F_1 = F(\lambda_1, \cdot)$ has at most a finite number of orbits that are either attractors or repellers, and it has no bifurcation orbits.*
(S_2) *F has PO-chaos at λ_1.*
(S_3) *Define $W = [\lambda_0, \lambda_1] \times \mathfrak{M}$. Then, there is a bounded subset of W that contains all periodic points in W.*

Then, there is a cascade in (λ_0, λ_1).

We hope at this point that the reader will feel that there are very few assumptions here and there does not seem that there are enough assumptions to prove anything. But from these mild assumptions, we assert that in the transition from no PO-chaos to PO-chaos, there must be a cascade. Continuation methods are strong enough for us to create a proof. In order to prove the existence of one cascade, we prove there are infinitely many cascades in W. Furthermore, each cascade of orbits is connected by a smooth curve of periodic orbits, as described in Sec. I A.

A. Components, cascades, and continuation

Definition 1 (Component). *A set $C \subset \mathbb{R} \times \mathfrak{M}$ is defined to be a* set of orbits *if $(\lambda, x) \in C$ implies x is a periodic point of $F(\lambda, \cdot)$; and in addition, every other point (λ, x') on the same orbit is also in C. We say that a set of orbits C is a* component *if it satisfies the following properties:*

(1) *C contains only regular orbits (meaning no flip orbits).*
(2) *C is path connected; i.e., for any two points (λ_a, a) and (λ_b, b) in C, there is a path of periodic points all of which are contained in C, such that one end of the path is (λ_a, a) and the other end is (λ_b, b). Furthermore, there is an upper bound on the periods of the periodic points on the connecting path. (The periods of orbits in a component are unbounded in the cases we consider).*
(3) *C is maximal, in the sense that it is not contained in a strictly bigger set of orbits satisfying (1) and (2).*

Note that each regular periodic orbit is contained in a component. By definition, flip orbits are not contained in components.

For generic F, each component C is either a closed loop of periodic orbits (where there is an upper bound on the periods of the orbits), or it contains a path

$$h(s) = (\lambda(s), x(s)) \quad \text{for } s \in \text{an open interval } J, \tag{2}$$

and the path $h(\cdot)$ passes through each orbit exactly once. In fact, we can always nonlinearly rescale s so that the open interval J is $(-1, 1)$, so we will always assume $J = (-1, 1)$. Note that $h(s)$ lies in each orbit of C for exactly one s. The function h is a called a "parametrization" of C.

Example 1 (A component). *In Fig. 3, we see a path $h(s) = (\lambda(s), x(s))$ for $s \in (-1, +1)$ that passes through a family of periodic orbits, leftmost of which is a saddle-node fixed point, and as we follow that to the right, the family undergoes a series of period-doubling bifurcations. Let C be the set of all (λ, x) that are points of orbits that h passes through. The set C is a component.*

We have already discussed cascades in a heuristic way, but in order to describe the details of our continuation methods, we now give a formal definition.

Definition 2 (Cascade). *A* cascade *is an infinite set of period-doubling bifurcation orbits O_n $n = 0, 1, \ldots$ such that for some k, each orbit O_n has period $2^n k$ with the following properties:*

(1) *There is a single component C that contains all the orbits O_n. Let $h(s) = (\lambda(s), x(s))$ for $s \in (-1, 1)$ be a parametrization of the component C and let $s_n \in (-1, 1)$ be the s values, so that each $h(s_n)$ is a point of O_n orbit.*
(2) *Either $s_n \to -1$ or $s_n \to +1$ as $n \to \infty$.*

We say the cascade is in the interval (λ_a, λ_b) if $\lambda(s_n) \in (\lambda_a, \lambda_b)$ *for all but finitely many n. We note that there could be two or more bifurcation orbits with the same period in a component C.*

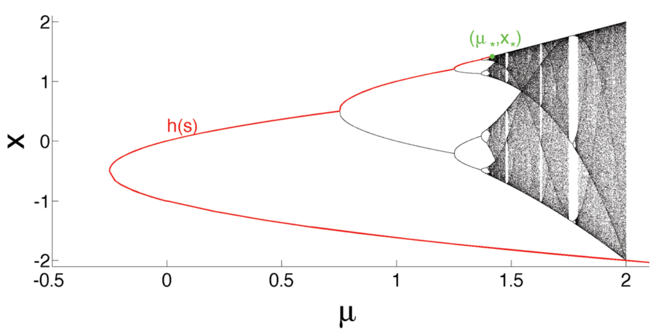

FIG. 3. *A path in a component.* There is one component that contains fixed points (as well as orbits of period 2^n for $n = 1, 2, 3, \ldots$. The path $h(s)$ shown in red passes through each orbit in this component exactly once. As shown, at each period-doubling bifurcation, $h(s)$ passes through the upper branch, but this choice is arbitrary. If the path were chosen so it sometimes went through the lower branch of periodic points, it would still pass through the same orbits. Both branches of periodic points describe the same orbits. Note that the lower branch extends infinitely far to the right and infinitely far down. The limiting end point (μ_*, x_*) is not a periodic point and is not a point of the cascade. The path can be thought of as the image of a function $h(s) = (\mu(s), x(s))$ for s in an open interval J, and the path $h(s)$ passes through each orbit exactly once. Furthermore, for an appropriate change of variables, we can choose $J = (-1, +1)$.

B. A comparison to previous results

There are at least two general approaches to the investigation of cascades, namely renormalization group methods and continuation methods.

Renormalization group methods. Renormalization group methods are analytical in nature, focusing on what happens near the limiting parameter value. The approach originated by Feigenbaum[23] and Coullet-Tresser,[19] with further work by Collet-Eckmann,[14–18] Sullivan[42,57,58] and the account in Ref. 20, Catsigeras,[10–12] and many others including[5,22,33,35–37] and references therein. The approach examines the parameter value where the event of interest occurs; in this case, the parameter value that is the limit of the period-doubling bifurcations and for generic situations describes what happens nearby (e.g., the parameters of consecutive period-doubling bifurcations scales in the well known way). This approach via the method of renormalization is extremely powerful and has lead to many deep dynamical systems results, including those leading to two recent Fields Medals. See Ref. 39 for a more complete description of the methods and historical context.

Continuation methods. The continuation approach is used in the current paper to describe behavior of generic systems. One examines the topological structure that causes the event that is being investigated, assuming one has a generic situation. The methods do not give scaling results such as seen with renormalization methods. However, they sometimes have the advantage of giving results for a larger class of systems. Continuation methods were used by Yorke-Alligood and Franks.[25,61] Franks used algebraic arguments arising from the Lefschetz index. Thus, his results do not give information on the nature of bifurcations. Our results extend the approach of Alligood and Yorke, which used a periodic orbit index. Our techniques are more general than those of Alligood and Yorke, as described below.

Two approaches to the implicit function theorem. To make the difference between the two approaches clearer, we describe the two corresponding approaches to a much simpler problem. Assume $G : R \times R^n \to R^n$ is a highly differentiable function. Let D be the open set of points (λ, x) for which the Jacobian matrix $D_x G(\lambda, x)$ is non-singular, and let $(\lambda_0, x_0) \in D$. The task is to describe the set of solutions of

$$G(\lambda, x) = G(\lambda_0, x_0), \tag{3}$$

that originate near (λ_0, x_0). Answering this question via an analytical approach gives the following.

Implicit function theorem: *On some sufficiently small interval $J = (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ centered at λ_0, there is a differentiable function X defined on J such that $X(\lambda_0) = x_0$ and $G(\lambda, X(\lambda)) = G(\lambda_0, x_0)$ for $\lambda \in J$.*

What happens to the set of solutions if we try to follow them beyond the small interval? A continuation approach addresses this question in the generic case. By making use of Sard's theorem, we get the following.

Global continuation theorem: *For almost every $(\lambda_0, x_0) \in D$, the set Y of solutions of (3) consists of one or more smooth curves. Let Y_0^0 denote the curve of solutions in Y that contains the point (λ_0, x_0).*

Then, Y_0^0 is a curve that does not cross itself. It is either a closed loop or is an unbounded curve.

If the curve is unbounded, then it is topologically equivalent to the real line, and $\lambda(s)^2 + |x(s)|^2$ must go to ∞ at both ends of the curve; that is Y_0^0 can be parameterized by the positive or negative of arc length s so that $Y_0^0 = \{(\lambda(s), x(s)) : s \in R\}$.

Then, $\lim_{|s|\to\infty}(\lambda(s)^2 + |x(s)|^2) = \infty$. In particular if the set of $(\lambda, x) \in Y$ with $|\lambda| \le \beta$ is bounded for every β, then points in Y_0^0 pass through all λ values.

See Refs. 2–4, 13, 40, 43, 44, and 49. Note the similarity of this theorem to the case of a component containing only one cascade.

Related continuation work. We end this introduction by briefly outlining what is contained in our previous papers on cascades and how they compare to the current paper.

The current paper, including Theorem 1, is a significant departure from the work of Yorke and Alligood,[61] who showed that if as a parameter changes, a map F develops a Smale horseshoe in a very specific manner, then there are cascades. In particular, they assume that there is a rectangle R whose image at one parameter value a does not intersect R but at another b maps twice across the rectangle so as to have the properties of a Smale horseshoe; we do not assume this (and our proof does not mention horseshoes). They also have further limitations which imply that between parameters a and b, there are no periodic points on the boundary of R. A perusal of Theorem 1 reveals that our work has no such assumption; we do not identify any rectangle R. In Theorem 1, we make no assumptions about how any horseshoe is created and instead make only mild assumptions about other orbits. Further, their work concluded that the (unperturbed) Hénon map has cascades (that lie inside R), whereas our approach can conclude that the Hénon map with large, generic, C^1-bounded perturbations has cascades.[51]

In Ref. 52, we use continuation methods to develop a more general n-dimensional version of our main lemma here that gives a criteria for the existence of a cascade for generic F. We apply those criteria here to two-dimensional systems depending on a parameter that develop PO-chaos. We note that we always prove there exists a cascade by proving there are infinitely many cascades.

We considered cascades for Hénon maps in Ref. 51. We developed the concept of periodic-orbit chaos and gave combinatorial results for counting cascades in Ref. 32. In Ref. 53, we gave a brief overview of many of our concepts and results, with the main focus on the characterization of periodic-orbit chaos and its relationship to positive topological entropy, as well as on the theoretical understanding of the generic Hopf bifurcations for maps. In particular, it announces without proofs versions of some of the results contained in the current paper: it contains images of numerical calculations for the double-well Duffing map and the forced-damped pendulum but only announces the cascades results in passing. In this paper, the double-well Duffing, Ikeda, and forced-damped pendulum equations are treated in detail in Examples 2, 3, and 4, and the discussion of coexisting sinks (not mentioned in the previous paper).

The paper proceeds as follows. In Sec. II, we discuss the implications of Theorem 1. In Sec. III, we apply our abstract theorem to show that a cascade exists.

II. CASCADES FROM BOUNDARIES

In this section, we give a more detailed discussion of the implications of Theorem 1. We start by discussing the notion of distance we use.

Definition 3 (Odd and even, exit and entry orbits). *Let F be generic and let p be a periodic point of a regular hyperbolic orbit for F. If p is a saddle, we call it either an odd orbit or an S orbit. If it is an attractor or a repeller, it is an even orbit or an AR. The odd-even terminology is due to the fact that the even points have an even number (either 0 or 2) eigenvalues that are greater than 1, while odd orbit has an odd number, namely 1, since it is a saddle. We mention the odd and even definitions only here to emphasize the nature of the distinction between AR and S orbits.*

If a regular orbit p is even, it is called an entry orbit for $\lambda = \lambda_0$; and for $\lambda = \lambda_1$, it is called an exit orbit. If the regular orbit p is odd, the reverse holds: it is called an entry orbit for $\lambda = \lambda_1$; and for $\lambda = \lambda_0$, it is called an exit orbit.

These definitions of entry and exit may seem rather arbitrary, but in fact they are derived from a specialized topological index known as the orbit index, first defined in Ref. 40, and it is this topological index theory that fundamentally underlies the proof of our theoretical results.

In Fig. 4, families of orbits enter the set $[\lambda_0, \lambda_1]$ at the saddles marked p_1, p_2, and p_3 and exit at p_4 and p_5. The family that enters at p_2 does not exit. The Global Continuation

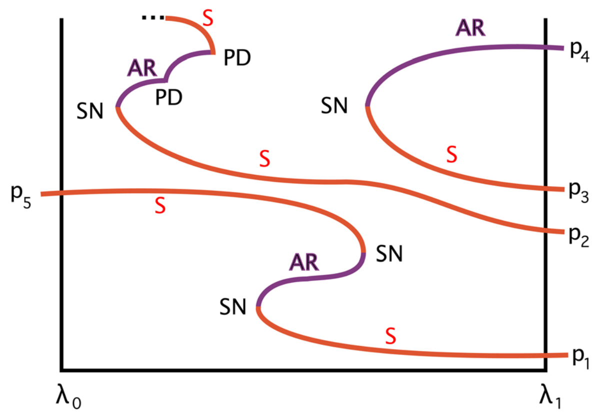

FIG. 4. *Paths on families of orbits*. Each point in the curves of this figure represents an entire orbit. This figure depicts three different paths of periodic orbits. It outlines the situation in Theorem 1 and Lemma 1. Path segments between bifurcations consist of regular saddles (denoted *S*), attracting orbits, or repelling orbits (both denoted *AR*). The bifurcation orbits for a generic family are all saddle-node (SN), period-doubling, or period-halving (both denoted *PD*) bifurcations. If while following a path, the path reverses direction ($\lambda(s)$ is increasing on one branch and decreasing on the other), as with p_3 and p_4 at *SN*, then one branch is Type *S* while the other is type *AR*. The path starting at p_1 reverses direction twice, both times at a *SN* point. If a path connects orbits at two ends of the parameter interval, as with p_1 and p_5, then they are the same type. We show that there are infinitely many Type *S* segments starting at λ_1 that do not return to λ_1, and only a finite number of these may connect to orbits at λ_0, and these must be Type *S*. Hence, there are infinitely many paths like the one stemming from p_2, remaining forever in a bounded region between λ_0 and λ_1 and containing cascades.

Theorem works because for a generic map, a family of solutions cannot suddenly stop, because it would have at least one limit point.

Theorem 1 declares the existence of a cascade. Below we describe the key lemma needed for the existence of cascades in proving Theorem 1. See Fig. 4. For a set W in parameter cross phase space, in keeping with our prior notation, we refer to a cascade as being in W if all but a finite number of bifurcation orbits of the cascade are in W.

Lemma 1 (Cascades from boundaries[52]). *Assume that F is generic and satisfies the boundedness condition (S_3) of Theorem 1. Recall that in (S_3), we define $W = [\lambda_0, \lambda_1] \times \mathfrak{M}$. Let C be a component parameterized by $h(s) = (\lambda(s), x(s))$ for $s \in (-1, 1)$. Then the following two statements hold:*

(i) (*The case of a path entering and exiting W.*) *Assume $-1 < s_1 < s_2 < 1$ are two consecutive values of s for which $\lambda(s)$ is on the boundary of the interval $[\lambda_0, \lambda_1]$. (That is, $\lambda(s_1)$ is equal to either λ_0 or λ_1, and $\lambda(s_2)$ is equal to either λ_0 or λ_1, but for all s so $s_1 < s < s_2$, we have that $\lambda_0 < \lambda(s) < \lambda_1$.) Then either $h(s_1)$ is an entry point and $h(s_2)$ is an exit point, or $h(s_1)$ is an exit point and $h(s_2)$ is an entry point.*

(ii) (*The case of a path with an end that does not leave W.*) *Assume that $-1 < s_1 < 1$ is such that $\lambda(s_1)$ is equal to either λ_0 or λ_1, but $\lambda_0 < \lambda(s) < \lambda_1$ for all $s \in [s_1, 1)$ or for all $s \in (-1, s_1]$. Then, C contains a cascade that is contained in (λ_0, λ_1).*

Sketch of proof of (i) of Lemma 1. Consider a component C and its parametrization $h(s) = (\lambda(s), x(s))$. Then, each segment of $h(s)$ between bifurcations consist of regular saddle points (denoted S), or attracting points or repelling points (both denoted AR). See Figure 4. Period-doubling and halving bifurcations only change the branch type when they also reverse the direction of $h(s)$ and due to genericity, there can only be a finite number of bifurcation points for $s \in [s_1, s_2]$.

When $\lambda(s)$ reverses direction, it must change type, i.e., from Type S to Type AR or vice versa. Specifically at a saddle-node bifurcation, the orbit type changes if and only if $\lambda(s)$ reverses direction, and the period of the two regular branches of the path differ by a factor of two. At a saddle-node bifurcation, a path changes, from a Type S branch to a Type AR or vice versa; the period of the two branches is the same.

If $\lambda(s_1) = \lambda(s_2)$, then the direction must have reversed an odd number of times, and the end orbits must be of opposite type, so one is an entry and one is an exit. If $\lambda(s_1) \neq \lambda(s_2)$, then an even number of reversals must have occurred and the end orbits are of the same type. Since the ends are on opposite ends of $[\lambda_0, \lambda_1]$, so one is an entry point and the other is an exit, as claimed in (*i*).

Sketch of proof of (ii) of Lemma 1. If $\lambda(s_1)$ is on the boundary of the parameter interval, and $\lambda_0 < \lambda(s) < \lambda_1$ for $s_1 < s < 1$, then let $Period(s)$ denote the period of the orbit at parameter s. (Note that the case $-1 < s < s_1$ is virtually identical to the case under consideration.) We claim $Period(s) \to \infty$ as $s \to 1$. Note that since $Period(s)$ can increase or decrease only in steps by a factor of 2 that would mean there must be an infinite number of such steps, and all of the periods $2^k \times Period(s_1)$ for all $k = 0, 1, 2, 3, \cdots$ must occur. Indeed, there must be period-doubling bifurcations with those periods, since as s changes, $Period(s)$ only changes at period-doubling bifurcation orbits. Hence, the component would contain a cascade.

To prove the claim, assume it is false. Then, there must be a sequence of parameter values $s_n \to 1$ as $n \to \infty$, for which $Period(s_n)$ is bounded. Since the periods are integers, there is a subsequence such that $p = Period(s_n)$ is constant. Since the sequence of points on the path is bounded in λ and x, it follows that there must be some limit point $z_0 = (\lambda, x)$ and since it is the limit of points of period p, it must be a periodic point and its period must divide p.

Since F is generic, the periodic point z_0 is on a path and any sequence of periodic points that limits on z_0 and are not in that path must have its periods going to ∞. Therefore, the sequence $h(s_n)$ must be on the path. Hence, the component ends at z_0, which is impossible, because the path through z_0 extends further, since by definition, we assume each path is maximal. Hence, we must have $Period(s) \to \infty$ as $s \to 1$.

Proof of Theorem 1. The proof that there is a cascade in Theorem 1 proceeds from the Lemma. The saddles in condition (S_2) are all entry orbits and they are infinite in number. There are only a finite number of exit orbits. Hence, there are infinitely many components that enter and do not exit. From the lemma, such a component has a cascade in the interval (λ_0, λ_1).

In light of Newhouse's work showing that infinitely many sinks can co-exist in chaotic regions, one might be concerned about the reasonableness of hypotheses (S_1) and (S_2) in Theorem 1: namely, we have required that at λ_1, all orbits are hyperbolic, there are infinitely many regular saddles, and finitely many attractors and repellers. It is conjectured that it is always possible to choose some λ_1 in the chaotic range, so that there are only finitely many attracting or repelling orbits at that value. Even though diffeomorphisms with infinitely many coexisting sinks are Baire generic (as Newhouse showed[47]), it is conjectured that they have "probability zero" in the sense of prevalence. See Gorodetski and Kaloshin[29] for recent partial results in this direction. The first results in this direction were much earlier in Refs. 59 and 48.

We may thus plausibly expect a generic parameterized map to have finitely many attractors for almost every parameter value. If this property is true, we can apply it to inverses of maps to conclude that there are only finitely many repellers for almost every parameter value. Hence, we can plausibly assume that for almost every parameter value, there are finitely many attractors and repellers. Even if the conjecture is false, our assumption is true for many systems.

III. APPLICATIONS AND OFF-ON-OFF CHAOS

Dynamical systems that satisfy conditions $(S_0), (S_2)$, and (S_3) for some λ_0 and λ_1 are plentiful. These give evidence of exhibiting PO-chaos, but a rigorous check of this condition is generally difficult at best and not practical. The following processes are examples that satisfy those three. We have chosen specific parameter values, though the phenomena described are seen over a wide range of parameter intervals.

Example 2 (The Ikeda map). *The Ikeda map models the field of a laser cavity.*[30] *For* $z \in \mathfrak{M} = \mathbb{C}$, *the complex plane, the map is*

$$F(\lambda, z) = \lambda + 0.9\, z\, e^{i\{0.4 - 6.0/(1+|z^2|)\}}.$$

At $\lambda = 0$, *there is a globally attracting fixed point. At* $\lambda = 1.0$, *we observe numerically a global chaotic attractor with a positive Lyapunov exponent and homoclinic points and one attracting fixed point and no repellers.*

Example 3 (The Pulsed Rotor map). *The Pulsed Rotor map with* $(x, y) \in \mathfrak{M} = S^1 \times \mathbb{R}$ *is*

$$F(x, y) = ((x + y)(mod\, 2\pi), 0.5y + \lambda \sin(x + y)).$$

For $\lambda = 0$, *there* is *a saddle fixed point and an attracting fixed point that attracts everything except for the stable manifold of the saddle. For* $\lambda = 10$, *we observe a chaotic attractor and a fixed point with a transverse homoclinic point.*

Geometric Off-On-Off chaos. We now define a class of parameterized maps that includes many nonlinear oscillators. Namely, there is no chaos at large and small λ values, but there is chaos at some intermediate value. We formalize these properties in the following definition:

Definition 4 (Off-on-off-chaos map). *Recall that the dimension of* $\mathfrak{M}$ *is* 2. *We say F is an off-on-off-chaos map if F is generic and satisfies the following properties:*

(D_0) *There are values* $\Lambda_1 < \Lambda_3$ *such that* $F_1 = F(\Lambda_1, \cdot)$ *and* $F_3 = F(\Lambda_3, \cdot)$ *each have at most a finite number of orbits.*
(D_1) *There is a* $\Lambda_2 \in (\Lambda_1, \Lambda_3)$ *for which* $F_2 = F(\Lambda_2, \cdot)$ *has at most a finite number of orbits that are attractors or repellers, and all of its orbits are hyperbolic.*
(D_2) F_2 *has infinitely many regular saddle orbits.*
(D_3) *There is a bounded subset of* $W = [\Lambda_1, \Lambda_3] \times \mathfrak{M}$ *that contains all of the orbits in W.*

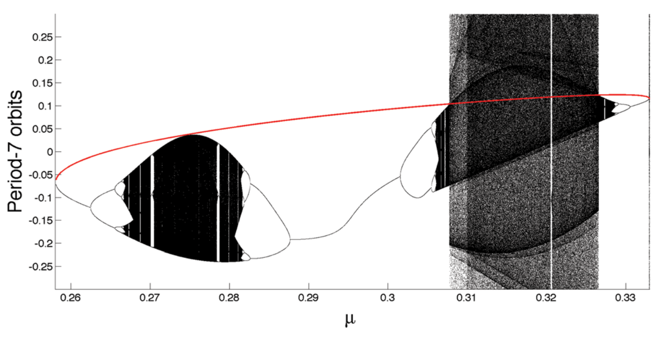

FIG. 5. *Paired cascades.* This figure depicts two sets of period-seven cascade components for the Hénon map $(u, v) \mapsto (1.25 - u^2 + \mu v, u)$, each containing paired cascades. Only one point of each of the period-7 orbits of the Hénon map are shown, so that it is clearer how each pair is connected by its component. The leftmost and rightmost cascades are paired; that is, they lie in the same component of orbits. They are connected by a path of unstable regular periodic orbits (the orbit connecting the lefthand and righthand sides of the figure). Likewise, the two middle cascades are paired. They lie on (and are connected by) a path of attracting period-seven orbits. Paired cascades are not robust to macroscopic changes in the map in that both can be simultaneously destroyed by a large enough local perturbation.

Although we cannot rigorously prove that there are parameter values for which chaos is observed and there are only finitely many sinks, it seems extremely likely to us that such parameter values exist for the double-well Duffing equation, forced damped pendulum, as well as other nonlinear oscillators not studied here. See Figure 2. Both are periodically forced with period 2π. Therefore, their time-2π maps, denoted by $F(\lambda, \cdot)$ or $F(\lambda, u, du/dt)$ are diffeomorphisms on $\mathbb{R}^2$.

The following definition distinguishes two types of cascades. See Figure 5.

Definition 5 (Paired and solitary cascades). *For a generic F, assume a component C has a cascade on each end. Then, we say the two cascades are paired. If a cascade is not paired, it is solitary.*

For a generic F, if there is only one cascade in component C on the set $W = [\Lambda_1, \Lambda_3] \times \mathfrak{M}$; then on one end, the component C continues without a limit in the interior of W. Thus, it must either be unbounded in x or reach the boundary of W. If assumption D_3 in the above definition holds, then x values in C are bounded, meaning that there is a point (λ, x) in C with either $\lambda = \Lambda_1$ or $\lambda = \Lambda_3$.

Theorem 2 (Off-on-off chaos). *Assume F is an off-on-off chaos map with* $\Lambda_1 < \Lambda_2 < \Lambda_3$ *given in the definition above. Let* $W = [\Lambda_1, \Lambda_3] \times \mathfrak{M}$. *Then, there is a cascade pair in W. Also there are at most finitely many solitary cascade components in W.*

As might be expected, we prove the existence of paired cascades by proving there are infinitely many such pairs.

Proof. This is a slight modification of the proof of Theorem 1. There are infinitely many regular orbits at Λ_2 but only finitely many have paths that extend to either Λ_1 or to Λ_3 or whose path returns to Λ_2. Therefore, there are infinitely many whose path (or component) remains in between Λ_1 and Λ_3 and has only one orbit at Λ_2. It follows that each of these has a cascade in (Λ_1, Λ_2) and another in (Λ_2, Λ_3); that is, the component has paired cascades.

If C is an unbounded cascade with some λ values in $[\Lambda_1, \Lambda_3]$, that is, it has some orbit in W, then it must have an orbit whose Λ coordinate is either Λ_1 or Λ_3. Since the map has only finitely many of these orbits, there can be at most finitely many such unbounded cascade components. □

Our numerical investigations strongly suggest that for the double-well Duffing Eq. (1), there are a number of intervals in the parameter range where F has a globally attracting fixed point. These intervals are centered near the values $\lambda \in \{1.8, 20, 73, 175, 350\}$.

Furthermore (2) between any two consecutive values in this set, there is a λ for which (some iterate of) the time-2π map appears to have a Smale horseshoe. Actually, it seems to have a chaotic attractor. Zakrzhevsky[63] provides many insights into the dynamics of a double-well Duffing equation though his version uses a restoring force of $u^3 + u$ instead of our choice of $u^3 - u$.

Example 4 (The forced damped pendulum). *The time-*2π *map for*

$$\frac{d^2\theta}{dt^2} + 0.3\,\frac{d\theta}{dt} + \sin\theta = \lambda \cos t$$

strongly appears to yield a geometric off-on-off-chaos map. We investigate the time-2π *map on* $\mathfrak{M} = S^1 \times \mathbb{R}$*; that is, the first variable is* $\theta\,(\mathrm{mod}\,2\pi)$ *and the second is* $d\theta/dt \in \mathbb{R}$*. There is a symmetry about* $\lambda = 0$*: For parameters* λ *and* $-\lambda$*, the system has the same dynamics. At* $\lambda = 0$*, there are only two periodic orbits, both fixed points, an attractor and a saddle, and we observe numerically a global chaotic attractor with a positive Lyapunov exponent and homoclinic points at* $\lambda = 2.5$*. For* $\lambda \geq 10$*, as at* $\lambda = 0$*, the two fixed points are the only orbits. Due to the friction term* $0.3\,d\theta/dt$*, the orbits must lie in a bounded subset of* $\mathfrak{M}$ *for* $\lambda \in [-10, 10]$*. This map then appears to be a geometric off-on-off-chaos map with either* $\Lambda_1 = 0$*,* $\Lambda_2 = 2.5$*, and* $\Lambda_3 = 10$*, or by symmetry, with* $\Lambda_3 = 0$*,* $\Lambda_2 = -2.5$*, and* $\Lambda_1 = -10$*. Assuming these numerical observations are valid, each cascade component must have its* λ *values lie entirely in either (0, 10) or (−10, 0). There are at most* $k = 4$ *unbounded cascade components and an infinite number of bounded pairs of cascades (where both are in the same component).*

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