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The Many Facets of Chaos

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There are many ways that a person can encounter chaos, such as through a time series from a lab experiment, a basin of attraction with fractal boundaries, a map with a crossing of stable and unstable manifolds, a fractal attractor, or in a system for which uncertainty doubles after some time period. These encounters appear so diverse, but the chaos is the same in all of the underlying systems; it is just observed in different ways. We describe these different types of chaos. We then give two conjectures about the types of dynamical behavior that is observable if one randomly picks out a dynamical system without searching for a specific property. In particular, we conjecture that from picking a system at random, one observes (1) only three types of basic invariant sets: periodic orbits, quasiperiodic orbits, and chaotic sets; and (2) that all the definitions of chaos are in agreement.

Keywords: Chaos; Lyapunov exponents; basins of attraction; horseshoes; forced-damped pendulum.

1. Introduction

Chaos is observed in so many ways that it can be quite confusing for a practitioner to get a reasonable answer to the simple question

"What is the definition of chaos?"

In fact, chaos cannot now be satisfactorily defined mathematically using a single definition, not because chaos is not a single concept, but because chaos has many manifestations in many different situations. There are situations in which many of these manifestations simultaneously apply, but each is independent of the others; while these definitions seem to agree in almost all cases, they tend to differ at the margins. In this paper, we discuss a variety of manifestations of chaos, the context where each would be likely to be found, and some of the methods used for detecting them. We restrict attention to deterministic dynamical systems, in which the initial conditions determine all future behavior. This paper is certainly not an attempt to be complete — there are many situations which we have left out altogether. We also freely admit that the particular focus is shaped by personal experience. These examples are meant to illustrate a point which we do believe is universal: For the definition to be useful, the determination of chaos should depend on the viewpoint of the investigator. It must be phrased in terms of the information that is available to the scientist in question.

The following is the list of aspects of chaos that we discuss, forming a summary of the rest of the paper:

- In Sec. 2, we discuss homoclinic orbits and horseshoes. These aspects of chaos are the earliest types of observed chaotic behavior, first described in the writings of Poincaré.
- Section 3 addresses strange chaotic attractors and their fractal topology.
- Section 4 describes sensitive dependence on initial conditions, including scrambled sets of Li–Yorke chaos, broad band power spectrum, and Lyapunov chaos. These latter two are commonly used in the study of time series data.
- Section 5 describes three nonequivalent types of entropy: topological entropy, periodic orbit entropy, and metric entropy.
- Section 6 covers important types of robust sets which are not attractors but may exhibit chaos: saddles and basin boundaries.
- Section 7 contains two conjectures on the types of behavior observable for a dynamical system selected at random. We give a formal definition for what we mean by observable. In broad strokes, we conjecture that for observable dynamical systems: the basic invariant sets have periodic, quasiperiodic behavior, or chaotic behavior; and multiple concepts of chaotic sets will agree for observable dynamical systems. In contrast to expected typical behavior, we list some commonly studied concepts which represent nontypical behavior.

2. Homoclinic Orbits and Horseshoes

If an orbit of a system containing an initial condition x has the same limit both in forward and backward time, x is called a *homoclinic point*. If the stable and unstable manifolds cross each other at a homoclinic point, then there are infinitely-many homoclinic points limiting on each other, and chaos occurs near this set of points. This situation is referred to as a *homoclinic tangle*. Figure 1 shows a *homoclinic bifurcation*, meaning the progression as a parameter is varied starting with no intersections between stable and unstable manifolds, followed by a tangency between manifolds, and ending with a homoclinic tangle. This was the type of chaos that Poincaré initially missed in his treatise on the stability of the solar system, which had won him a prize given by King Oscar III of Sweden. Poincaré assumed that there were no homoclinic crossings. When Phragmén pointed out his error, his attempts at a correction resulted in his creating the modern field of dynamical systems [McGehee, personal communication]. In particular, he clearly realized the complex nature of the situation in the following description of the situation depicted in the bottom two panels of Fig. 1 [Poincaré, 1993, pp. I59]: "If one seeks to visualize the pattern formed by these two curves and their infinite number of intersections, each corresponding to a doubly asymptotic solution, these intersections form a kind of lattice-work, a weave, a chain-link network of infinitely fine mesh; each of the two curves can never cross itself, but it must fold back on itself in a very complicated way so as to recross all the chain-links an infinite number of times. One will be struck by the complexity of this figure, which I am not even attempting to draw."

More than 60 years after Poincaré's work, Smale [1967] rigorously described homoclinic tangles by showing that the behavior of homoclinic tangles is at least as complicated as the behavior of a very simple iterated map that he called a horseshoe for its geometric shape. In particular, he was then able to fully define all dynamical behavior under the horseshoe map. In particular, he showed that although the map is deterministic, the behavior of the orbits is as random as a coin flip, includes infinitely many periodic orbits, and displays exponential divergence of trajectories. The set of points with these hallmarks of chaos (known as the chaotic set) also has an interesting fractal structure. The chaos in this case can be *transient*, meaning that the chaotic set is unstable. After creating the seemingly arbitrary map, Smale demonstrated that for every fixed point (or periodic orbit) with a homoclinic crossing, there is a horseshoe map embedded in the behavior of the system. Therefore, any system with a homoclinic crossing includes the type of random behavior described above. The relationship between homoclinic points and chaos motivates much of the study of homoclinic points and their bifurcations.

Verifying chaos by checking for homoclinic crossings is in some cases feasible when other manifestations — such as power spectrum or Lyapunov exponent — cannot be used. This method has even been used in infinite-dimensional contexts, such as for parabolic partial differential equations and delay differential equations [Lani-Wayda & Walther, 1995, 1996]. (More general results in this

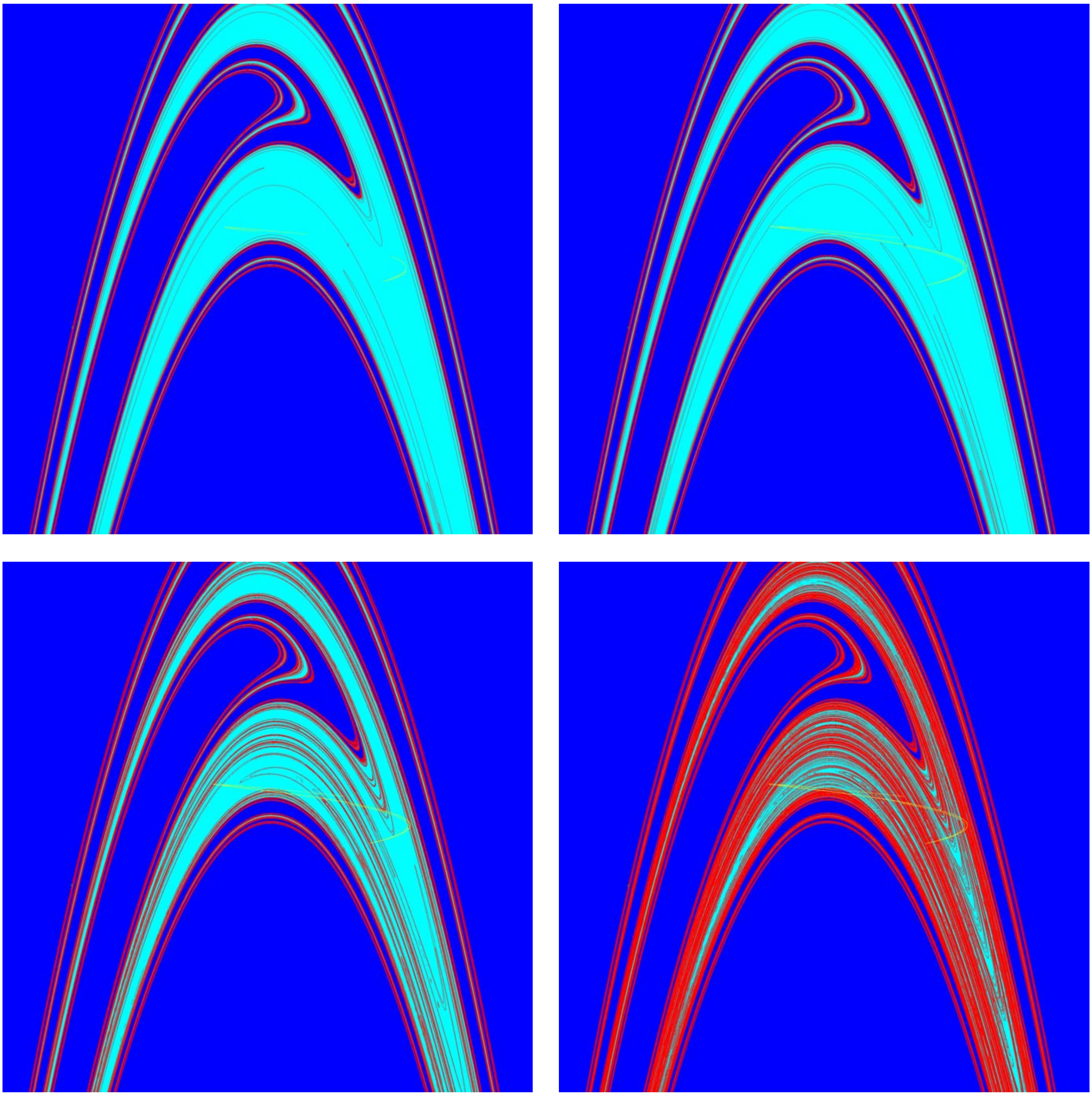

Fig. 1. The formation of a homoclinic tangle in the Hénon map: $(x, y) \mapsto (\rho - x^2 - 0.3y, x)$ for $-4 < x < 4, -3 < y < 3$, left to right, up to down $\rho = 2.0, 2.01725, 2.01875, 2.0246$. The stable manifold for a fixed point is shown in red. The unstable manifold contains the attractor, in yellow. In the first figure, the stable and unstable manifolds do not cross and the attractor is disconnected. In the second figure, the stable and unstable manifolds are tangent, and the attractor becomes connected. The manifolds cross in the third and fourth figures, becoming a tangled mess limiting on themselves infinitely often. Homoclinic tangles are one of the best known behaviors associated with chaos.

spirit show the existence of Shilnikov heteroclinic orbits for delay equations [Lani-Wayda, 2001].) Although the horseshoe is most often used to study homoclinic tangles, some dynamical systems can be shown to have a horseshoe directly without first showing the existence of a homoclinic orbit. This is, for example, true for Hénon maps [Devaney & Nitecki, 1979] and in some rigorous computational proofs of chaotic behavior [Arai & Mischaikow, 2006]. There are shortcomings of testing for a chaotic set via chaotic orbits. As mentioned above, the set may be unstable, referred to as transient chaos. In addition, the existence of a homoclinic crossing is not a quantitative measure of either the amount of chaos in a system or its robustness under variation of a parameter.

3. Chaotic Attractors and Their Topology

Homoclinic crossings are not the aspect of chaos that Ueda so most prominently observed when he first viewed the Duffing map attractor in 1961 [Hayashi *et al.*, 1970], depicted in Fig. 2. Rather, he was struck by the repeatability and irregular topology of the attractor. Similar observations of robustness and geometric complexity were made by Edward Lorenz for the well known Lorenz attractor [Lorenz, 1963].

The fractal nature of attractors is a signature of chaos — and attractors with this property are referred to as *strange attractors*. (It is also possible to have a strange nonchaotic attractor with fractal structure, but these structures are nontypical and highly degenerate[1] [Grebogi *et al.*, 1984].) To say that an attractor "looks fractal" has been made rigorous using a variety of attractor dimension definitions, and it is numerically feasible to check a variety of dimensions for an attractor for a low-dimensional system, though in attractor of higher dimensions these calculations become far more complex and in some cases infeasible.

Even with this more sophisticated mathematical machinery, there is still no clear definition of a strange attractor. However, there is one special case of strange attractors called *rank-one attractors* that are well defined and relatively well understood [Young, 2003; Wang & Young, 1999, 2001, 2013]. Figures 1–3 show examples of the relationship between strange attractors and homoclinic tangles. In particular, in Figs. 1 and 3, one sees that topological changes in the attractor occur at homoclinic bifurcations: two attractors merge into a single attractor when stable and unstable manifolds are tangent at a homoclinic bifurcation.

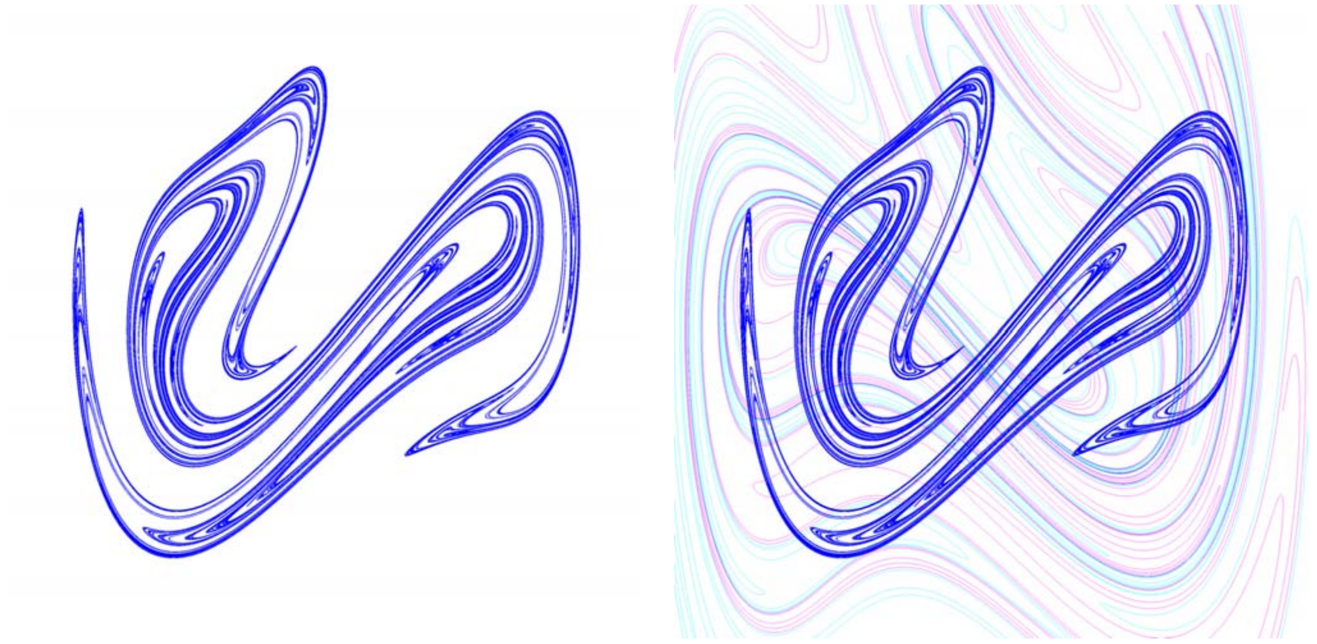

Fig. 2. The attracting set for Ueda's forced double-well Duffing map: $x''(t) + 0.05x'(t) + x(t)^3 = 7.5\sin(t)$ for $-2.2 < x < 2.2$, $-1.5 < y < 2.6$. The solution is plotted every 2π (stroboscopic), which defines an invertible map. Looking at this attractor in 1961, Ueda observed the repeatability of the irregular (now known to be fractal) topology [Hayashi *et al.*, 1970]. This is one of the signatures of chaotic sets. (Left) The attractor is plotted alone. (Right) The Ueda attractor plotted together with a portion of the two branches of the stable manifolds (in magenta and cyan) of a saddle fixed point at the origin, thus showing that there is a homoclinic tangle. If all of the stable manifold branches were plotted, they would densely cover the attractor and the region surrounding it.

[1]It is possible that strange nonchaotic attractors are of infinite codimension. This is discussed further in Sec. 7.

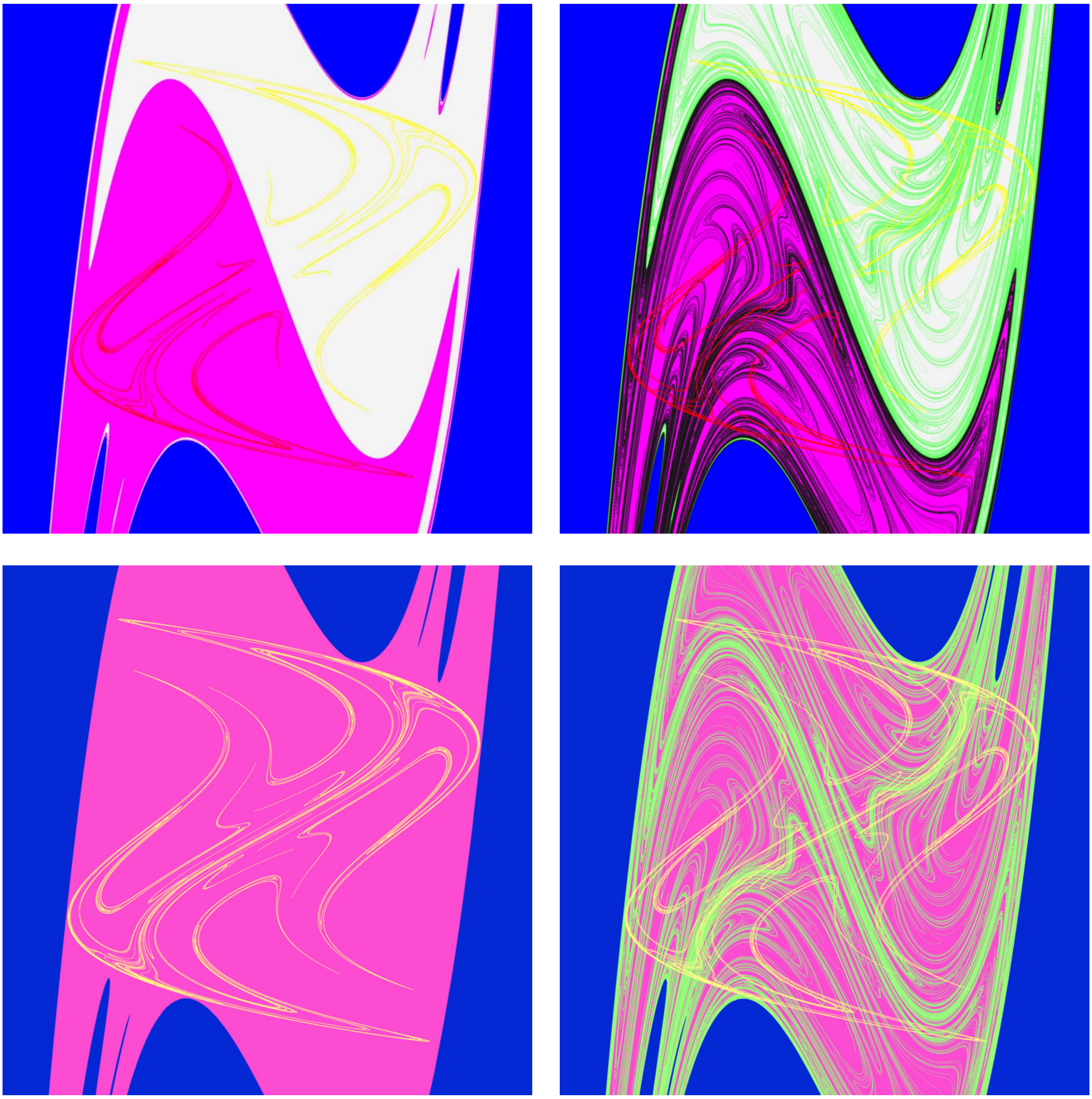

Fig. 3. Chaotic attractors for the Holmes map: $(x, y) \mapsto (1.5x - x^3 + \lambda y, x)$ for $-2 < x < 2, -2 < y < 2$. There are three fixed points, one at the origin and two outer ones satisfying $x = y = \pm\sqrt{0.5 + \lambda}$ for $\lambda > -0.5$. In the upper two pictures $\lambda = 0.7$ and there are two attractors (yellow and red) whose basins are white and magenta. In the lower two figures, $\lambda = 0.714$, the attractors have merged. In all four pictures, blue represents the basin of ∞ and at the center of the picture $(0, 0)$ is a fixed point whose stable manifold in the upper two figures is the boundary between the white and magenta. For the two figures on left, only the attractors and the basins are shown. The upper right figure includes the stable manifolds of outer fixed points in black and green while in the lower right, those manifolds (green) have crossed the stable manifold and their closures are equal. If plotted to infinite extent, it appears that those stable manifolds would come arbitrarily close to every point in the basin of the attractor. The merging of attractors occurs between 0.7 and 0.714 when the stable and unstable manifolds of $(0, 0)$ cross, creating new homoclinic points.

4. Sensitive Dependence on Initial Conditions

Perhaps the easiest chaotic system to understand analytically is the *doubling map* f on $[0, 1]$ where

$$f(x) = 2x \pmod 1,$$

that is $f(x)$ is the noninteger part of $2x$. The point 0 is a steady state, since $0 = 1 \pmod 1$. The point $x = 1/3$ exhibits periodic behavior since $f(1/3) = 2/3$, and $f(2/3) = 4/3 \pmod 1 = 1/3$. However, points arbitrarily close to 1/3 deviate from the orbit of 1/3, where the deviation becomes exponentially large with the iterate. For example, the point 0.33 is close to 1/3, but $f(0.33) = 2 \times 0.33 = 0.66$, $f^2(0.33) = 0.32$, the fractional part of 1.32. The deviation on the seventh iterate is more than 100 times as great as the original deviation. Further, the deviation is larger than 1/3: the distance between the two points in the original orbit. In general, for initial points x, y that are sufficiently close together, when we apply the map n times to each we get $|f^n(x) - f^n(y)| = 2^n|x - y|$, provided that $|x - y| \leq 0.5 \times 2^{-n}$. Table 1 shows this deviation. This example demonstrates one aspect of chaos: exponential divergence of trajectories, also referred to as *sensitive dependence on initial conditions*.

Another aspect of chaos demonstrated by this example is an infinite number of periodic orbits. We have shown by example that 1/3 is a periodic point. In fact, the map f has infinitely many periodic points. For example, for each prime number p and integer $0 < m < p$, m/p is a periodic point; i.e. $f^{p-1}(m/p) = m/p$. This follows from the so-called Fermat's Little Theorem [Weisstein, 2014], which states that for each integer k (here we are interested in $k = 2$) and each prime p, $k^{p-1} - 1$ is an integer multiple of p. Therefore, for each integer m, $m \times k^{p-1} - m$ is an integer multiple of p. Dividing by p yields $(m/p) \times k^{p-1} - m/p = 0 \pmod 1$. Hence $f^{p-1}(m/p) = m/p$ so m/p is a periodic orbit and its period divides $p - 1$. Note that $f^3(1/7) = 1/7 \pmod 1$ whereas $1/3, 1/5, 1/11$, and 1/13 have periods $2, 4, 10$ and 12 respectively. It is however easy to see that this map has at most a finite number of period k points for each k, so there is no bound on the periods of the periodic orbits.

4.1. *Li–Yorke chaos*

In 1975, Li and Yorke [1975] reported on sensitive dependence on initial conditions in their definition of a chaotic set for a map F. Such a set was later referred to as a *scrambled set*. The definition requires both that the set is mixing in the sense that for any two distinct points p and q in the set, the distance between $F^n(p)$ and $F^n(q)$ occasionally get arbitrarily close to each other but then move apart and get closer together and go apart, coming arbitrarily close but not staying close. An uncountable scrambled set with an infinite number of periodic orbits has Li–Yorke chaos. See also Sec. 5, where we discuss entropy — topological entropy is an alternative codification of mixing, and periodic orbit entropy is an alternative codification of a large number of periodic points, but all of these concepts are distinct. Li–Yorke has proved to be useful in the study of maps, but it is rather hard to render numerically or verify rigorously, especially for higher-dimensional maps and flows. The next two concepts in this section are much more useful for data analysis.

4.2. *Broad power spectrum chaos*

Gollub and Swinney observed chaotic motion in Taylor–Couette flow fluid experiments, but they had no analytical representation of the system [Gollub & Swinney, 1975]. Rather, their data was in the form of an experimental time series, and their indicators of chaos were the broad power spectrum for the time series data. This is the basis of the numerical *0–1 test* for chaos [Gottwald & Melbourne, 2009a, 2009b; Melbourne & Gottwald, 2008]. The broad power spectrum only considers the behavior of orbits, ignoring nearby trajectories, thus missing the geometric aspects of the chaos.

Table 1. Iteration of the doubling map.

Iterate	0	1	2	3	4	5	6	7
Point	1/3	2/3	1/3	2/3	1/3	2/3	1/3	2/3
Point	0.33	0.66	0.32	0.64	0.28	0.56	0.12	0.24
Deviation	1/300	2/300	4/300	8/3007	16/300	32/300	64/300	128/300

An idea which does consider nearby trajectories is Lyapunov chaos.

4.3. *Lyapunov chaos*

By far the most common method of checking for chaos in maps and flows is by using *Lyapunov exponents*, a quantitative measure of the degree of stretching or contraction of the system along an orbit. A positive exponent implies the existence of a direction for which the system is stretching along the orbit. *Lyapunov chaos* occurs if there is a positive probability of a random trajectory having an expanding direction. It is a useful technique as it gives quantitative information about the degree of chaos: the degree of stretching and the number of stretching directions. This information can be used to make estimates on the fractal nature of the chaotic set. In some cases, there is a relationship between Lyapunov exponents and power spectra [Slipantschuk *et al.*, 2013]. A comparison of numerical methods can be found in [Barrio *et al.*, 2009].

4.4. *Finite time Lyapunov exponents*

They are used for time series data using delay coordinate embedding and an attractor reconstruction, recovering information about the map by considering its topological properties. In fact, an entire branch of research has sprung up using finite time Lyapunov exponents to analyze data [Shadden *et al.*, 2005; Shadden, 2005].

A full attractor reconstruction may not work well in the presence of noise. For example, in [Mytkowicz *et al.*, 2009] the computer is shown to be a chaotic dynamical system in terms of its cache behavior, resulting in a reconstruction of a 12-dimensional attractor. However, the presence of noise in the system implies that a two-dimensional projection gives much more reliable results.

Results on spurious Lyapunov exponents [Tempkin & Yorke, 2007] show a simple example in which calculation of Lyapunov exponents for a time series gives completely misleading information regarding the original system. The paper illustrates spurious Lyapunov exponents with an example of a five-dimensional time series of the deterministic Hénon attractor where the original Hénon attractor has Lyapunov exponents $\alpha > 0$ and $\beta < 0$, but the time series has Lyapunov exponents $2\alpha, \alpha, \alpha+\beta, \beta, 2\beta$. Thus even in this simple example, the largest and smallest exponents do not serve as a quantitative measure of the expansion and contraction rates for the original system. The results call into serious question whether Lyapunov exponents for time series data give any sort of meaningful quantitative measurement of the original system. We conjecture that the following weaker statement holds: if the time series data has a positive exponent, then the original system does as well.

Note that it is difficult to distinguish deterministic chaos from noise. However, if our attractor reconstruction yields a successful result, it is the result of deterministic behavior rather than of random fluctuations.

5. Three Nonequivalent Entropies

Chaos for a finite-dimensional map is often characterized as possessing positive entropy. In this section, we point out that there are (at least) three distinct nonequivalent definitions of entropy. Furthermore, there are concrete contexts in which each is the superior definition, in that the other two are impossible to verify.

5.1. *Topological entropy*

For a dynamical system, the *topological entropy* measures the mixing of the domain. Specifically, orbits of length n are said to be *distinguishable* with resolution ϵ if they vary by more than ϵ in the first n iterates. The topological entropy is defined as the exponential growth of the number of mutually distinguishable orbits as the resolution goes to zero. Specifically, let $N(n, \epsilon)$ be the number of distinguishable orbits with length n. Then the entropy is the limit as the resolution ϵ goes to zero of $h(\epsilon)$, where $h(\epsilon) = \limsup_{n\to\infty} \log N(n, \epsilon)/n$.

Recall from Sec. 4 that a scrambled set also has a form of mixing. In fact, positive topological entropy implies that there exists an uncountable scrambled set [Blanchard *et al.*, 2002]. However, there are cases with an uncountable scrambled set with zero topological entropy [Misiurewicz & Szlenk, 1980].

Positive topological entropy is a useful concept in the case of continuous maps which are not necessarily differentiable. It is not useful in the case where one is only provided with a trajectory as the output of an experiment. It is also not easily computed numerically, though it has been proved to be positive using rigorous computational methods in some cases [Day *et al.*, 2008; Newhouse *et al.*, 2008; Frongillo & Treviño, 2012].

5.2. *Periodic orbit entropy*

For a map f, if the number of periodic orbits grows exponentially with the period, we say the map has *periodic orbit chaos*. That is, the supremum of the number S_p of fixed points of f^p grows exponentially with respect to p. This growth factor $\rho = \limsup_{p\to\infty} \log S_p/p$ is the *periodic orbit entropy*. Periodic orbit chaos corresponds to a positive periodic orbit entropy.

One application of periodic orbit chaos is in understanding the transition from order to chaos. *Period-doubling cascades* consist of an infinite number of period-doubling bifurcations as a parameter is varied (cf. Fig. 4). They have long been noted as a hallmark of the transition from order to chaos, in contexts as varied as quadratic maps, ordinary differential equations, infinite-dimensional systems such as partial differential equations, delay equations such as the one shown in Fig. 4, and experimental data. In [Sander & Yorke, 2012, 2013], we were able to show the exact relationship between cascades and the transition to periodic orbit chaos. We show that if a typical[2] family of two-dimensional maps with bounded periodic orbits transitions from a finite number of periodic orbits to an infinite number of saddles and a finite number of attractors and repellers, then the transition contains period-doubling cascades.

Positive periodic orbit entropy is equivalent to positive topological or metric entropy except in special cases. There are many works dedicated to showing exactly when these concepts are equivalent. The next paragraph gives a brief journey through some of these results.

Bowen showed that for Axiom A diffeomorphisms, periodic orbit chaos is equivalent to positive topological entropy [Bowen, 1970]. Wang and Young proved that the same is true for Hénon-like maps. However, Chung and Hirayama [2003] showed that for maps on surfaces with Hölder continuous derivatives, positive topological entropy occurs exactly when there is exponential growth of saddle periodic points (ignoring attracting and repelling periodic points). For interval maps, the periodic and topological entropies may not be equivalent [Chung, 2001; Katok & Mezhirov, 1998]. The number of periodic orbits can grow significantly faster than exponentially — corresponding to infinite periodic-orbit entropy. Artin and Mazur proved that maps with at most exponential growth of periodic orbits are dense in the space of diffeomorphisms. However, periodic orbit growth can be superexponential, growing like $e^{n^{1+\epsilon}}$ as n increases for some $\epsilon > 0$ and in particular bigger than topological entropy for maps in one and higher dimensions [Kaloshin & Kozlovski, 2011]. Kaloshin and Hunt showed that for $r \geq 2$ this is nongeneric. That is, a generic

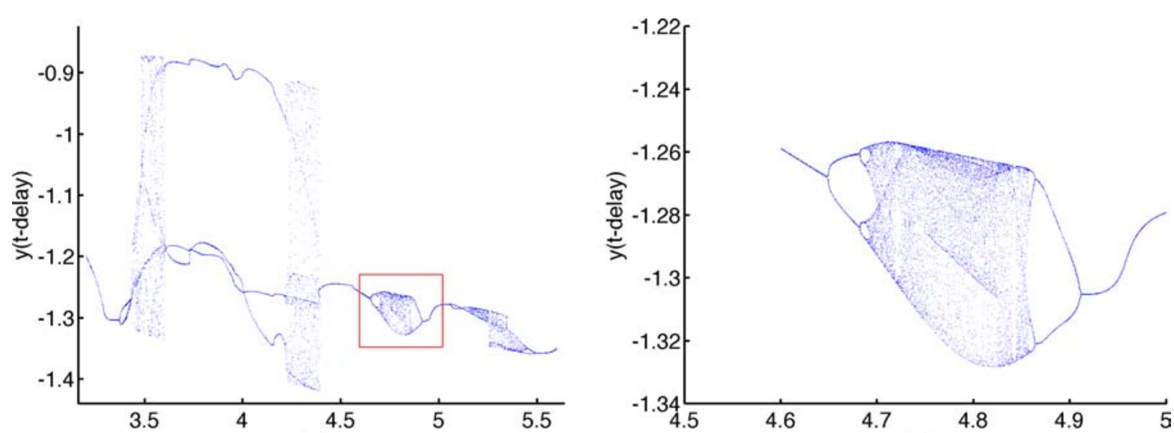

Fig. 4. Period-doubling cascades occur for the delay equation $dx/dt(t) = F(x(t-\tau)) = -2x(t-\tau)e^{-x(t-\tau)^2}$, where delay $= \tau$ is the parameter for the horizontal axis. This graph is made by creating a series of trajectories, each using fixed τ, and plotting $(\tau, x(t-\tau))$ whenever t is such that $x(t) = 0$ and $dx/dt > 0$.

[2]More technically, the statement is true for a generic set of map families, meaning that it is true for an arbitrarily small perturbation of any family.

subset of an open set of maps has superexponential growth of periodic orbits, but the growth is not much faster than exponential for a prevalent set [Kaloshin & Hunt, 2001a, 2001b]. Generically, this behavior occurs in a homoclinic class [Bonatti *et al.*, 2008]: Every nonhyperbolic homoclinic class of a C^1 generic diffeomorphism with a finite number of homoclinic classes has superexponential growth of the number of periodic points. For a hyperbolic homoclinic class the entropy is equal to the periodic orbit entropy. If a set is topologically mixing, then there are constants so that the entropy bounds the periodic orbit growth. Generic C^1 diffeomorphisms in the complement of the Axiom A plus no cycle diffeomorphisms have superexponential growth, whereas the topological entropy of any diffeomorphism of a compact manifold is finite.

5.3. *Metric entropy*

Whereas topological entropy looks at mixing rates for all orbits, *metric entropy* measures mixing probabilistically, looking at the degree to which the dynamical system mixes typical orbits. Topological entropy is an upper bound on metric entropy.[3] The Pesin entropy formula relates metric entropy to the Lyapunov exponents: for smooth maps that preserve Borel measure, the metric entropy is the sum of the positive Lyapunov exponents. Ruelle showed that with no assumption on the metric, this becomes an inequality, i.e. the sum of the positive Lyapunov exponents is an upper bound for the metric entropy. Ledrappier and Young made further contributions to showing the precise relationship between Lyapunov exponents and metric entropy. See [Young, 2003]. These results do not apply for general infinite-dimensional equations or for time series.

6. Robust Chaotic Sets that are Not Attractors

In this section, we discuss instants of chaos that are robust under parameter changes, but are not attractors.

6.1. *Chaotic saddles*

The term *chaotic saddle* refers to an invariant chaotic set S that is unstable; that is, arbitrarily close to the set there are initial points whose trajectories leave some neighborhood of S. (The term saddle is often used even if the set is a repeller.) Above the word chaotic can be used in any of the ways we have discussed, whether in terms of broad power spectrum, Lyapunov exponents, Li–Yorke chaos (for maps), or homoclinic points. A chaotic saddle is usually a fractal. The invariant set of the horseshoe map is a simple example of such a set. Anytime there is a homoclinic tangle, there are horseshoes, so unless the horseshoe is included in a larger attractor, the invariant set will be a chaotic saddle. Chaotic saddles are robust in the sense that they persist under sufficiently small changes in the dynamical system. Figure 5 shows a chaotic saddle for the Holmes map $(x, y) \mapsto (1.5x - x^3 + \lambda y, x)$ for multiple λ values.

6.2. *Chaotic basin boundaries*

The boundary between basins of attraction of two or more attractors can contain a chaotic set, in which case there exist special case of chaotic saddles, often with fractal structure. Even when the attractors themselves have simple behavior, the boundary between basins can contain chaotic sets. Figure 6 shows examples of chaotic attractors and chaotic sets in boundaries of basins of attractors in the case of forced-damped pendulum equation. See also our movie showing the full sequence of basins and boundaries for $\rho \in [1.5, 3.5]$ [Sander & Yorke, 2014].

7. Observable Behaviors in Dynamical Systems

In this section, we aim to put chaos in context. In trying to describe chaos, one might ask "chaos — as opposed to what?" That is, what other basic building blocks besides chaotic sets are a dynamical system likely to have. In particular, besides chaotic sets, for a randomly chosen dynamical system, not selected for any particular kind of behavior, what kinds of basic invariant sets are likely to exist? We start by making the phrase "randomly chosen" more precise. We then present a short list of behaviors which we conjecture to be observable. We then describe some examples of special dynamical systems that have been finely tuned to have some special property, a property we would not encounter in a randomly chosen dynamical system.

[3] In fact, the topological entropy is the supremum of the metric entropies taken over all f-invariant Borel probability measures.

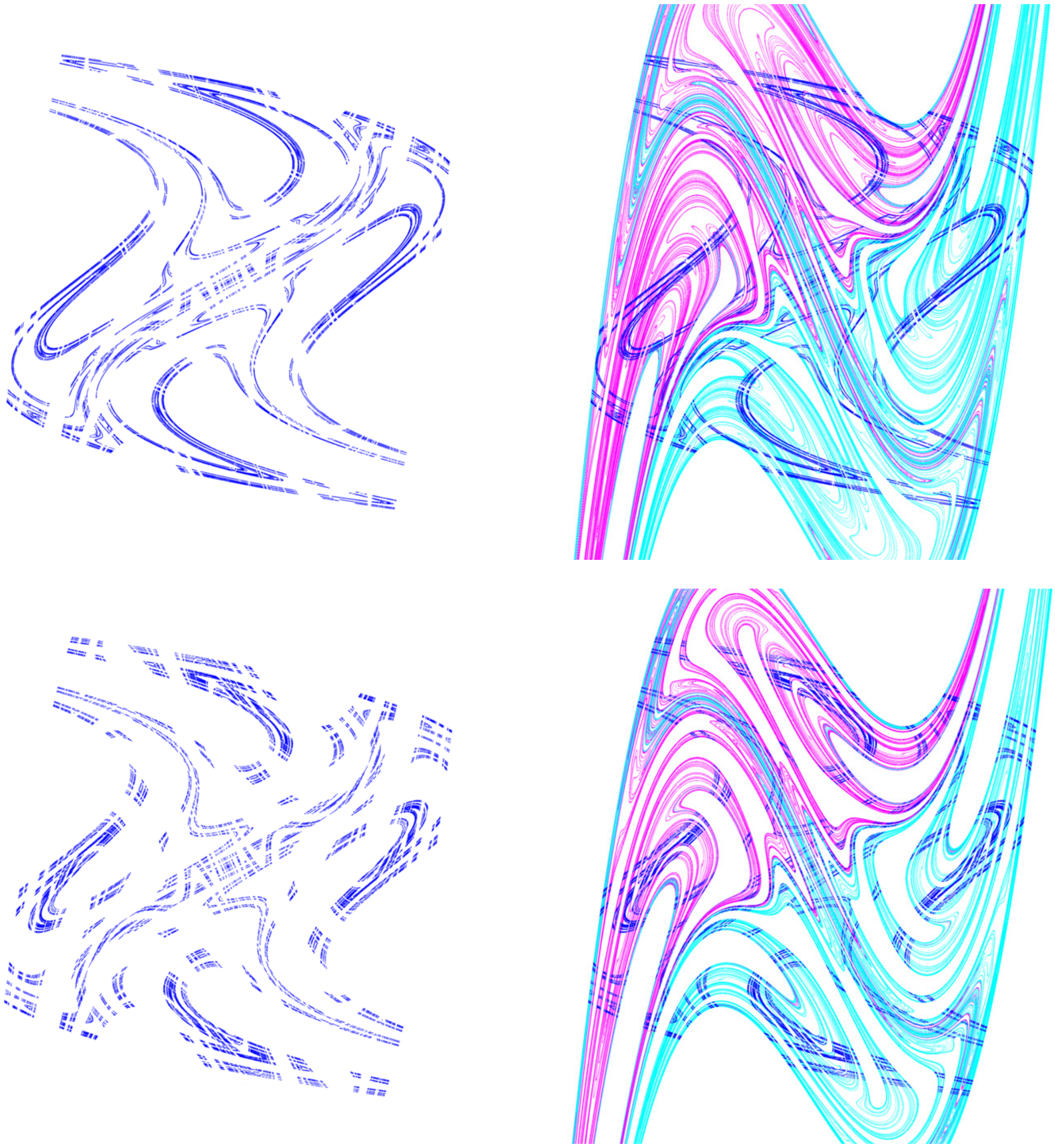

Fig. 5. A chaotic saddle set for the Holmes map: $(x, y) \mapsto (1.5x - x^3 + \lambda y, x)$ for $-2 < x < 2, -2 < y < 2$, where from up to down $\lambda = 0.8, 0.95$. The saddle is in blue (left and right). On the right, the two branches of the stable manifold for the fixed point at $(0, 0)$ are shown in magenta and cyan. This set is invariant and chaotic though it is not stable under either the map or its inverse. It is however robust with respect to a change in parameter. The fractal nature of the set is apparent here.

7.1. *Typical, observable, and rare behaviors*

If a dynamical system is picked at random, we would like to know what dynamical behaviors we should expect to see. We distinguish three concepts: (i) *typical*: behavior we expect for almost every randomly chosen dynamical system, (ii) *observable*: behavior we expect to occur with positive probability, and (iii) *rare*: behavior that occurs with probability zero. To be more specific as to where we take our probability measure for these concepts, in the context of a general dynamical system, either a map or a differential equation, imagine a huge set of smooth dynamical systems depending on several parameters, and imagine that we could use a perfect random number generator to pick the parameters at random to select one dynamical system. The precise technical details involve the idea of prevalence.

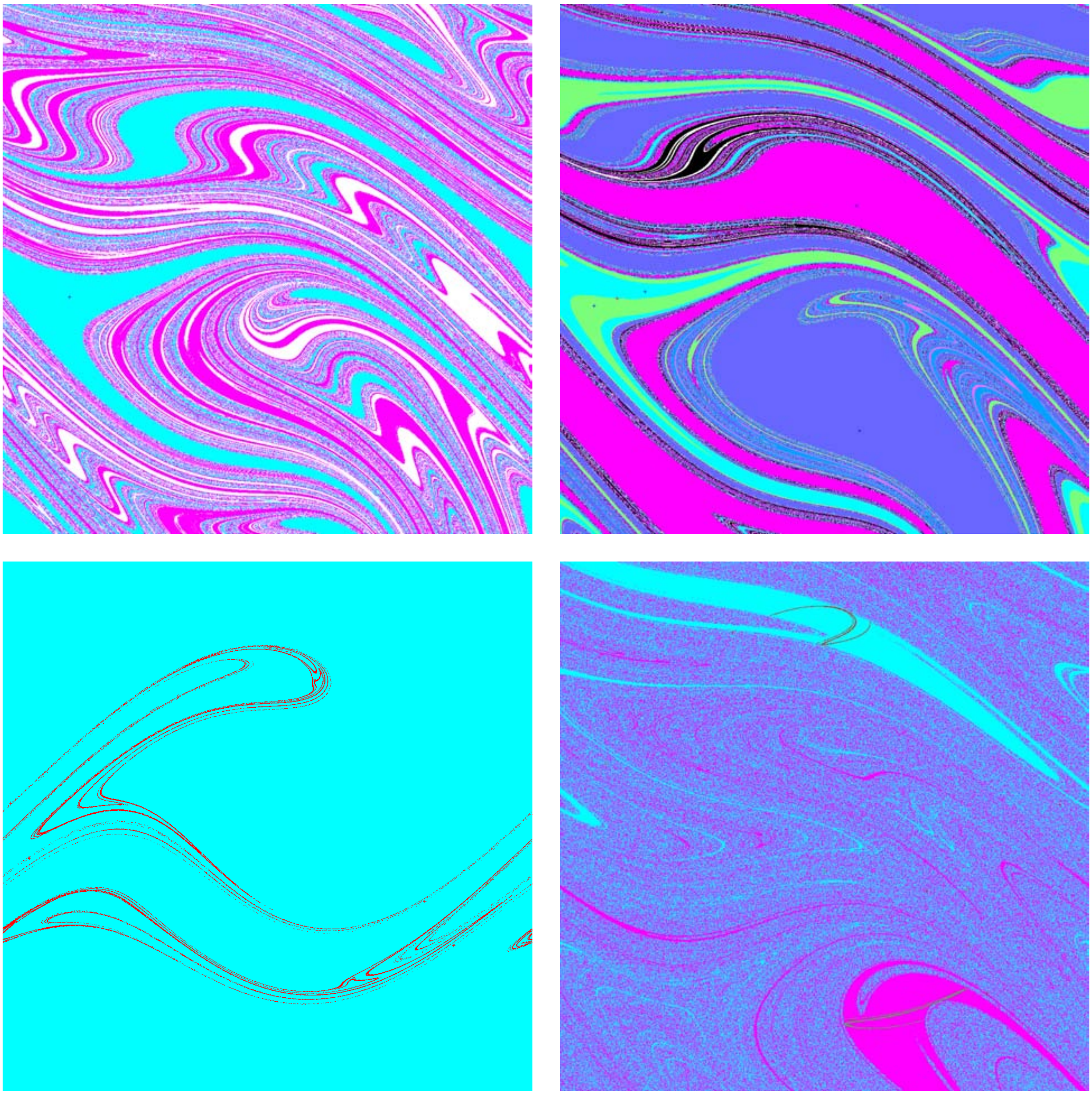

Fig. 6. Fractal basins of attraction for the forced-damped pendulum: $x'' + 0.2x' + \sin x = \rho \cos t$, for $-\pi < x < \pi, 2 < y < 4$, and $\rho = 1.5725, 1.73, 2.3225, 3.0875$ left to right and up to down. The solution is plotted every 2π (stroboscopic), defining an invertible map. The third figure demonstrates a globally attracting chaotic attractor. The first and second show fractal boundary for the many basins of attraction for attractors with simple behavior: periodic orbits. In the second figure, there are, in fact, eight distinct but intermingled basins of attraction. The fourth figure shows a fractal boundary of a basin of attraction of a chaotic orbit. For a more complete picture, see our movie showing the entire sequence of changes of basins and boundaries for $\rho \in [1.5, 3.5]$ [Sander & Yorke, 2014].

See, for example [Hunt *et al.*, 1992; Ott & Yorke, 2005; Kaloshin, 1997].

Note that in above we are assuming to have a specific dynamical system: if the map depends on parameters, then we are talking about the probability of a specific choice of parameters. Thus bifurcation points can be rare behaviors. Such a statement has already been rigorously proved in a number of cases, especially for periodic orbits: It is shown in [Hunt *et al.*, 1992] that for almost every map (in the sense of prevalence), there are no saddle-node or period-doubling bifurcation periodic orbits. Of course, these orbits can occur in parametric families of maps, but the parameters for which they occur have measure or probability zero. To show that a behavior is rare, it is not sufficient to study a single parametric family. For example, for the logistic map $rx(1-x)$, the set of r that are the limit of an infinite number of period doublings (i.e. the Feigenbaum points) appears to be measure zero in the set of randomly chosen parameter values r. However, even if we show that Feigenbaum points are measure zero in r, we could not conclude without further work that they are rare in the sense of prevalence. To show that the behavior is rare, we would need to consider the behavior in high-dimensional parameter spaces.

7.2. *Basic sets*

The term *basic set* refers to any of the basic invariant building blocks of dynamical systems. Chaotic sets and isolated periodic orbits (including isolated fixed points or steady states) can all be examples of basic sets. Specifically, for a differential equation or map on the n-dimensional space $\mathbb{R}^n$ (or on a smooth surface) we define a *basic set* B to be a set that is closed, has a dense orbit, and is *maximal*, meaning that there is no strictly larger closed set with a dense orbit containing B. That is, there is no point x in the space whose positive limit set strictly contains B. In particular, if a chaotic set is a basic set, then no periodic orbit contained in that chaotic set is a basic set because that periodic orbit would not be maximal.

There are three kinds of sets of particular interest that can be basic sets.

7.3. *Periodic basic set*

The simplest type of basic set is a periodic orbit which is maximal. The basic set consists of a finite number of points.

7.4. *Chaotic basic set*

We call a basic set with chaotic behavior using any of the definitions described in this paper a chaotic basic set.

7.5. *Quasiperiodic basic set*

In addition to chaotic and periodic dynamics, one observes *quasiperiodic* dynamics. For example, Fig. 7 shows what happens near a parameter value for which there is a Hopf bifurcation for a map, that is, a Neimark–Sacker–Hopf bifurcation. For each $a \in [19, 22]$ there is an invariant closed curve — which is topologically a circle — and on each circle there is either a periodic behavior or the map is equivalent (under some nonlinear change of variables) to an irrational rotation. In the latter case, the topological circle is a quasiperiodic basic set.

Such a map could also arise as a Poincaré return map of a periodically forced differential equation, (like a periodically forced van der Pol equation) in

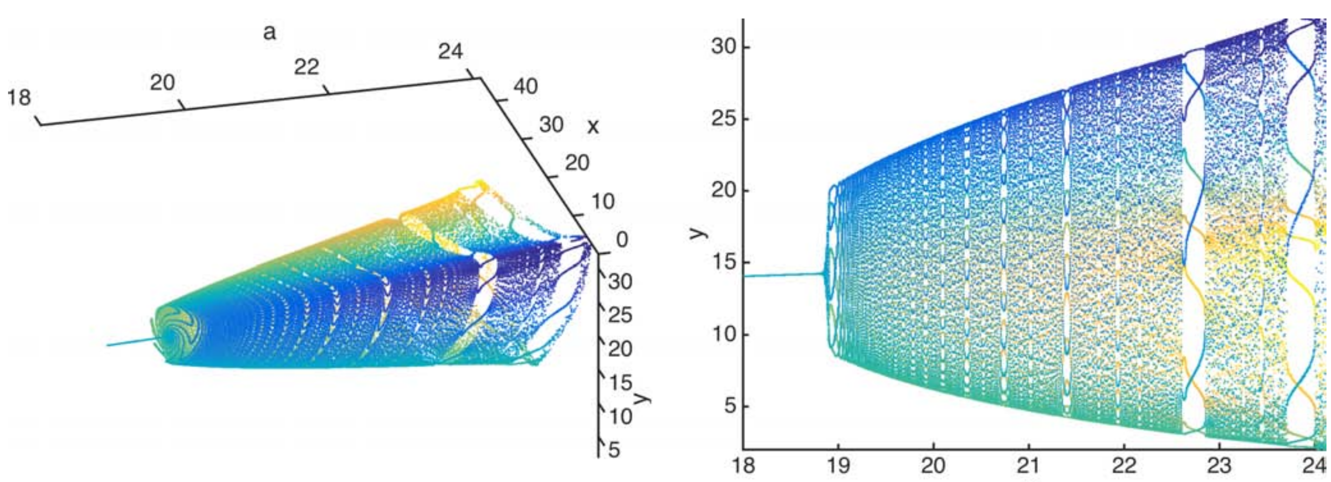

Fig. 7. Two views of a Neimark–Sacker–Hopf bifurcation for an overcompensatory Leslie map $T(x,y) = (a(x+y)e^{-0.1(x+y)}, 0.7x)$, where $18 < a < 24$ [Ugarcovici & Weiss, 2004]. Near the (subcritical) bifurcation, there is an attracting invariant circle. Around every parameter a for which the rotation of the circle is rational, there is a region with an attracting periodic orbit instead of an attracting invariant circle, resulting in the holes on the surface.

which case the topological circle is a cross-section of a topological torus.

We can investigate the behavior of these topological circles by restricting attention to maps of a circle. In 1961, Arnold [2009] reported on the analytic maps on a circle such as that by

$$\theta_{n+1} = \theta_n + \omega + \frac{K}{2\pi}\sin(2\pi\theta_n) \pmod 1,$$

where all θ_n are in $[0, 1)$ as are ω and K. We think of θ as lying on a circle and ω is a rotation of that circle while the final term is a nonlinear perturbation of this rotation. The map has a periodic attractor for ω in infinitely many different disjoint closed intervals in $[0, 1]$, but Arnold found that the complementary set C of ω is also large in the sense that C has positive measure, and the closure of C is a positive measure Cantor set (also known as a *fat* Cantor set). Hence if ω is chosen at random, there is a positive probability that ω is in C. For such ω, the limit set of each θ_1 is the entire interval [0, 1]. He showed that for such cases, there is a change of variables that converts that map into an irrational rotation of the circle. For such ω, we say the map is *quasiperiodic*.

The quasiperiodic case more generally includes maps and flows on higher-dimensional topological tori for which there is a change of variables into a system that is a rotation on each of $k \geq 1$ circles,

$$\theta_{1,n+1} = \theta_{1,n} + \omega_{1,n}, \ldots, \theta_{k,n+1} = \theta_{k,n} + \omega_{k,n}.$$

7.6. *Observable basic sets*

The following conjecture states that the three basic sets given above is a complete set of what is observable.

Conjecture 1 (Three Observable Types of Basic Sets). *The following are the only types of basic sets which are observable*:

(C) *a maximal chaotic set*
(P) *a periodic orbit which is maximal*
(T) *a topological torus* (*on which trajectories are quasiperiodic*). *For maps, this includes invariant topological circles on which there are no periodic orbits.*

7.7. *Typical chaos*

We have outlined a variety of definitions of chaos, emphasizing that there may not always be perfect agreement between the definitions of what is chaotic, but the definitions appear to agree most of the time and we conjecture below that this is typically the case.

Conjecture 2 (Typical Agreement of Chaos Definitions). *For a typical smooth dynamical system, the sets which are chaotic by one of the definitions listed below are chaotic for each of the other definitions listed below.* (*This is actually a separate conjecture for each pair of definitions.*)

(E1) *Positive topological entropy.*
(E2) *Positive periodic orbit entropy.*
(E3) *Positive metric entropy.*
(L) *Positive Lyapunov exponent.*
(H1) *Containing homoclinic orbits.*
(H2) *Containing horseshoes* (*for some iterate of the map or large time of the flow*).

For one-dimensional maps, the definitions of horseshoes and homoclinic orbits must be correctly interpreted, cf. [Sander & Yorke, 2009]. We do not expect that the numerical values for the three entropies to be typically the same, merely that they will have the same sign.

7.8. *Examples of rare dynamics*

In contrast to our discussion of observable behaviors, there are dynamical systems that have a large literature but occur only seldom. We list a few of these unlikely but important types of dynamical behaviors, all of which we conjecture to be rare in the sense of prevalence.

- Transitional behavior such as chaotic crises where as a parameter is varied a chaotic attractor changes suddenly, possibly disappearing and other bifurcation behaviors;
- Strange nonchaotic attractors and more generally quasiperiodically forced systems;
- Higher-dimensional attracting invariant tori;
- Interval exchange maps (which can occur as Poincaré return maps of specially tuned dynamical systems);
- A Feigenbaum point: the limit of a cascade of period doublings in a system depending on a parameter.

8. Conclusion

The effort to define chaos is reminiscent of the discussion over the planethood (and eventual lack thereof) of the dwarf planet Pluto, as can be seen in the following comments from [Brown, 2010]: "... scientists work by concepts rather than definitions. ... Nature abhors a definition. Try to lock something into too small a box and I guarantee nature will find an exception. A planet is something that is big and important in the solar system, something that dominates its part of the solar system, but the actual definition becomes murky at the margins." Likewise there are many different hallmarks of chaos, which all work well in many cases for many people, but to try to pin chaos to one formal definition that works for all mathematicians, scientists, and engineers is to ignore the nature of nature itself.

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