The Many Facets of Chaos

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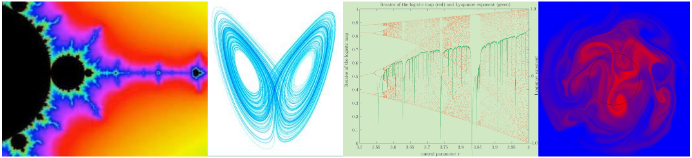
Definitions of Chaos



"Chaos" ala Google

- Typical dynamical systems either have simple trajectories such as steady state and quasiperiodic orbits, or they exhibit chaos.
- Chaos is defined in so many ways that it is quite confusing for a practitioner to get a reasonable answer to the simple question

  "What is the definition of chaos?"

- We assert that this is the wrong question to ask

Definitions of Chaos

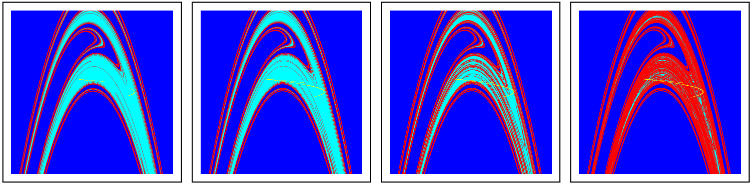
Buddist parable of the three blind monks and the elephant

- Chaos cannot be satisfactorily defined mathematically using a single definition, not because chaos is not a single concept, but because chaos has many manifestations in many different situations.

- In this talk: A variety of manifestations of chaos, with the conjecture that typically the different forms of chaos are equivalent.

- Disclaimer: While this illustrates a universal point, the specific list is incomplete, shaped by our personal knowledge and experience

Earliest Observations: Transverse Homoclinic Orbits



Stable and unstable manifolds of fixed point, Hénon map
$(x, y) \mapsto (\rho - x^2 - 0.3y, x)$, $-4 < x < 4, -3 < y < 3$,
$\rho = 2.0, 2.01725, 2.01875, 2.0246$

- Poincaré initially thought all homoclinic orbits coincided when they intersected.
- Phragmén pointed out his error.

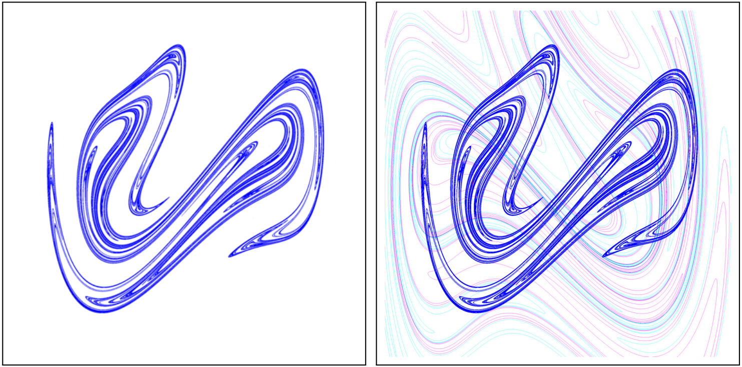
Transverse Homoclinic Orbits: Poincaré

- Poincaré, 1892: "If one seeks to visualize the pattern formed by these two curves and their infinite number of intersections, each corresponding to a doubly asymptotic solution, these intersections form a kind of lattice-work, a weave, a chain-link network of infinitely fine mesh; each of the two curves can never cross itself, but it must fold back on itself in a very complicated way so as to recross all the chain-links an infinite number of times. One will be struck by the complexity of this figure, which I am not even attempting to draw."
- Movie

Transverse Homoclinic Orbits: From Smale to Present

- Smale, 1967: Horseshoe maps are contained in transverse homoclinic orbits, implying chaos.
- Useful characterization when visualization is difficult but analysis is tractable, such as delay equations, PDEs.
- Often gives rise to transient chaos, not attracting
- Not quantitative

Ueda 1961, Lorenz 1963: Robustness and Irregular Topology



Ueda-Duffing map: $x''(t) + 0.05x'(t) + x(t)^3 = 7.5\sin(t)$,
$-2.2 < x < 2.2, -1.5 < y < 2.6$, 2π- stroboscopic

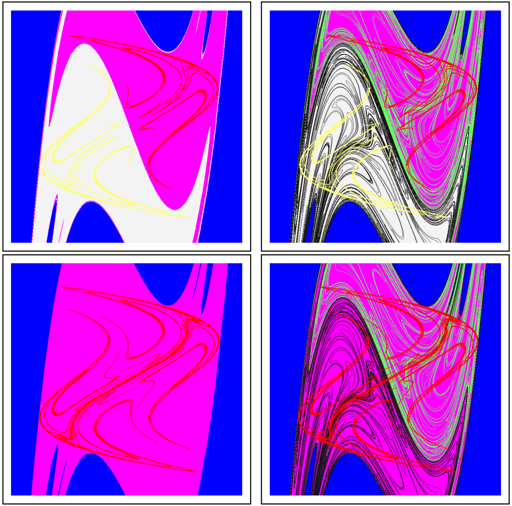
Right, stable manifold branches

Lorenz made similar observations

Chaotic Attractors

- An eyeball measure of fractal topology in an attractor of a low-dimensional system is an easy method of chaos detection.
- This has since been made more precise in the form of attractor dimension calculations
- There is no clear definition of strange attractors – rank one attractors are one special subtype [Wang and Young]
- In fact, homoclinic points play a role...

Homoclinic Orbits and Merging of Chaotic Attractors



Holmes map: $(x, y) \mapsto (1.5x + x^3 + \lambda y, x)$,
$-2 < x < 2, -2 < y < 2$, $\lambda = .8$ (up), $.9$ (down) .

Chaos in Time Series: Power Spectrum

- Gollub and Swinney 1975 observed chaotic motion in Taylor-Couette flow fluid experiments
- No underlying map
- Data in the form of a time series
- Indicators were based on the broad power spectrum for the time series data
- This method only considers behavior of orbits, ignoring nearby trajectories

Exponential Divergence of Trajectories

$$x \mapsto 2x \mod 1$$

It	0	1	2	3	4	5	6	7
x_i	1/3	2/3	1/3	2/3	1/3	2/3	1/3	2/3
$\overline{x}_i$	0.33	.66	0.32	0.64	0.28	0.56	0.12	0.24
Er	0.0033	.0067	0.133	0.0267	0.0533	0.1067	0.2133	0.4267

- Exponential divergence of trajectories used in definition of scrambled sets
- Generalizes to positive stretching along solutions, meaning positive Lyapunov exponent
- Lyapunov chaos: Positive probability of a random trajectory having an expanding direction.

Lyapunov Chaos

- Common method of checking for chaos for maps and flows
- Gives quantitative measure of degree of chaos
- Finite time Lyapunov exponents are used for time series data using delay coordinate embeddings and attractor reconstruction
- In cases of noisy data, a full attractor reconstruction may give less reliable results than a two-dimensional projection. [Mytowicz, Diwan, Bradley, Computer cache data, 09]
- Spurious Lyapunov exponents can occur in time series, with no easy way to distinguish the true exponents [Sauer, Tempkin, Yorke, 98]
- Methods such as the $0-1$ test attempt to avoid reconstruction [Gottwald, Melbourne]

Periodic Orbit Chaos

- For map f, let S_p be the number of fixed points of f^p. Periodic orbit chaos means positive periodic orbit entropy:

$$\limsup_{p \to \infty} \frac{\log S_p}{p}$$

- We used periodic orbit chaos to show results on period-doubling cascades [Sander and Yorke, 09–13]
- Without theoretical methods, computation cannot be made rigorous

Positive Topological Entropy

- Topological entropy measures the mixing of a set. For $N(n, \varepsilon)$ the distinguishable orbits, topological entropy:

$$\lim_{\varepsilon \to 0} \lim \sup_{n \to \infty} \frac{\log N(n, \varepsilon)}{n}$$

- Positive topological entropy is a concept of chaos useful in the case of continuous, analytically defined, not necessarily smooth maps.
- It is not easily numerically computable, though has been used for rigorous computational proofs [Day et al. and Newhouse et al. 08, Frongillo 14]

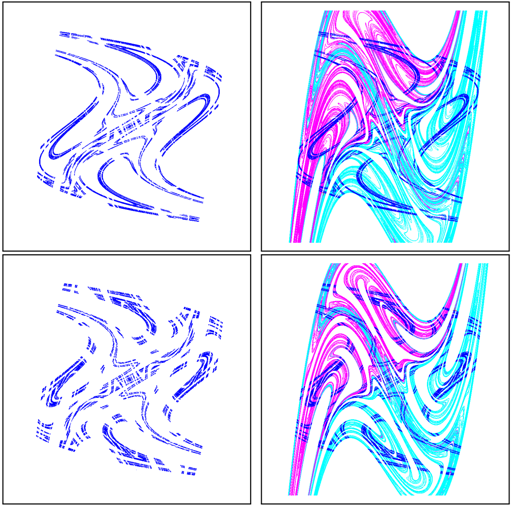
Comparing to Entropies

- Metric entropy is related: Topological entropy is an upper bound on metric entropy
- In finite dimensional maps and flows topological entropy for smooth maps is equal to the sum of the positive Lyapunov exponents when a measure is SRB. Otherwise, the difference is in terms of the dimension of the invariant measure [Pesin, Ruelle, Margulis, Ledrappier/Young 85].
- The relationship is unknown for general infinite dimensional equations and time series.

Comparing Entropies

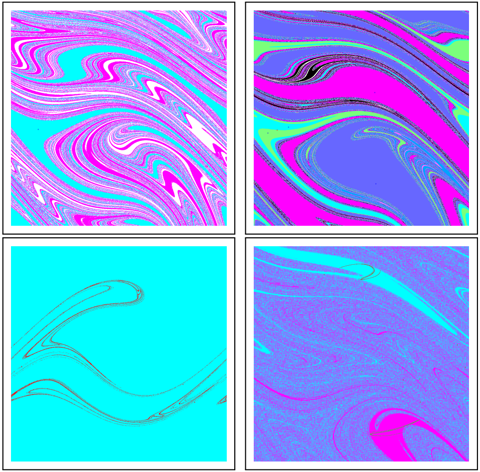
- Periodic orbit chaos is equivalent to positive topological entropy for Axiom A diffeomorphisms [Bowen, 70] and Hénon-like maps [Wang and Young]. There can be superexponential growth of periodic orbits [Kaloshin]. Thus the concepts are in general not equivalent
- There are zero topological entropy sets which are scrambled sets.
- No clear relationship between positive Lyapunov exponents and decay of time correlations [Slipantschuk, Bandtlow, Just, 13].

Chaotic Saddles, Robust But Not Stable



Holmes map: $(x, y) \mapsto (1.5x + x^3 + \lambda y, x)$,
$-2 < x < 2, -2 < y < 2, \ \lambda = 0.8, 0.95$

Fractal Basins



Forced-damped pendulum:
$x'' + 0.2x' + \sin x = \rho \cos t$,
$-\pi < x < \pi, 2 < y < 4$,
$\rho =$
$1.5725, 1.73, 2.3225, 3.0875$,
2π-stroboscopic

- Attractors globally attracting (3), multiple basins (1,2,4)
- Fractal basin boundaries (1,2)
- Eight distinct basins (2)
- Chaotic attractors (3,4)
- Movie

Two Conjectures

What happens typically? Typical means either a generic or prevalent set.

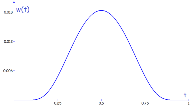
Chaos Conjecture: For a typical smooth dynamical system, the definitions of chaos are equivalent: three types of entropy, positive Lyapunov exponent, transverse homoclinic orbits, horseshoes.

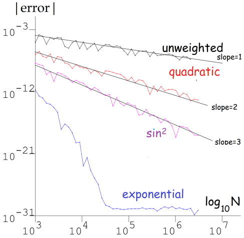
Typical Behavior Conjecture: Consider a basic set: maximal compact invariant transitive set for a map or flow in finite dimensions. For a typical set of equations, if this set does not have positive Lyapunov exponent, then the set is a steady state, periodic orbit, or quasiperiodic set.
Quasiperiodic means topological circle or torus of some dimension (for maps, multiple tori).

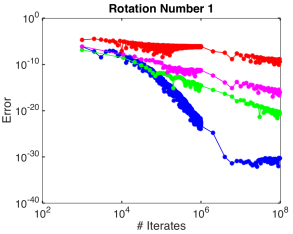
Current work: Quasiperiodicity

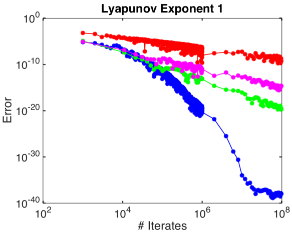
- Fast accurate method for calculating Lyapunov exponent, rotation number, Fourier coefficients for a quasiperiodic map.

- The method resembles windowing methods in signal processing.

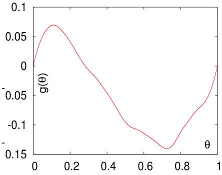


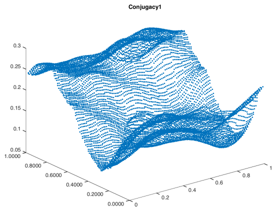


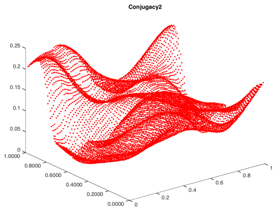


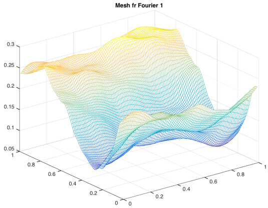


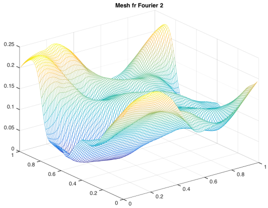
Conjugacy in 1D and 2D



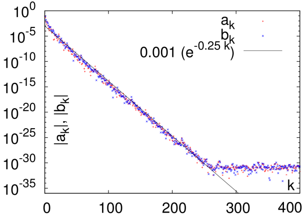


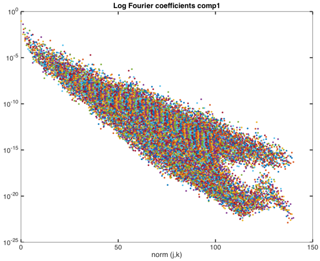


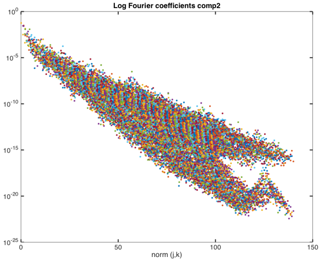




Fourier coefficients decay in 1 and 2 dimensions







Outlook

- Each definition of chaos comes with its own strengths and shortcomings – both numerical [Barrio, Borczyk, Breiter 07] and theoretical
- The concept is too big for a definition – no one mathematical definition will suffice
- "Scientists work by concepts rather than definitions ... Nature abhors a definition try to lock something into too small a box and I guarantee nature will find an exception." -*Discover Magazine*, in reference to Pluto and the demise of its planethood.